Convergence in law of the maximum of the two-dimensional discrete Gaussian free field

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Abstract

We consider the two-dimensional Gaussian Free Field on a box of side length N, with Dirichlet boundary data, and prove the convergence of the law of the recentered maximum of the field.

1 Introduction

The discrete Gaussian free field (GFF) $\{\eta_{v,N} : v \in V_N\}$, on a box $V_N \subset \mathbb{Z}^2$ of side length N with Dirichlet boundary data, is the mean zero Gaussian process that takes the value 0 on ∂V_N and satisfies the following Markov field condition for all $v \in V_N \setminus \partial V_N$: $\eta_{v,N}$ is distributed as a Gaussian random variable with variance 1, and mean equal to the average over its immediate neighbors given the GFF on $V_N \setminus \{v\}$. (For convenience, we will take V_N to have its lower left corner at the origin.) One aspect of the GFF that has received intense attention recently is the behavior of its maximum $\eta_N^* = \max_{v\in V_N} \eta_{v,N}$. Of greatest relevance to this paper are the papers [3], where it is proved that $\eta_N^*/(2\sqrt{2/\pi}\log N) \to 1$ in probability; [5], where it is proved that, for

$$m_N = 2\sqrt{2/\pi}\big(\log N - \tfrac{3}{8}\log\log N\big), \tag{1}$$

$\mathbb{E}\eta_N^* = m_N + O(1)$ and $\eta_N^* - \mathbb{E}\eta_N^*$ is a tight sequence of random variables; and [8], where rough asymptotics of the probability $\mathbb{P}(\eta_N^* \geqslant m_N + x)$ are derived for large x. Aspects of this model have been treated in both the mathematics and physics literature; we refer the reader to [8] for an extensive discussion of the history of this problem.

Once it has been established that the fluctuations of $M_N := \eta_N^* - m_N$ are of order 1, it is natural to study the convergence of the laws of M_N, in particular, whether the laws of M_N do indeed converge. Our goal in the current paper is to establish this convergence, as stated in the following theorem.

Theorem 1.1. *The law of the random variable $\eta_N^* - m_N$ converges in distribution to a law μ_∞ as $N \to \infty$.*

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We will also provide a description of the limit law μ_∞ in Theorem 2.3 of Section 2.

Besides the intrinsic interest in the study of the GFF, we note that it is an example of a logarithmically correlated model. The behavior of the maxima for such models is conjectured to be universal; see, e.g., [6] for (non-rigorous) arguments using a renormalization-group approach and links to the freezing transition of spin-glass systems, and [11] for further information on extreme distributions. On the mathematical side, numerous results and conjectures have been formulated for such models; see [9] for recent progress. Theorem 1.1 above provides a partial answer to [9, Conjecture 12].

The proof of Theorem 1.1 can be described roughly as follows. We fix a large integer K, partition V_N into K^2 boxes of side length N/K, and introduce a Gaussian field X_v^f that we refer to as the *fine field*, and which roughly corresponds to the value of the GFF minus its conditional expectation given the sigma-algebra generated by the GFF on the boundary of these sub-boxes. The fine field has the advantage that, due to the Markov property of the GFF, its values in disjoint boxes are independent. We define the *coarse field* as the difference $X_v^c = \eta_{v,N} - X_v^f$. The coarse field is, of course, correlated over the whole box V_N, but it is relatively smooth; in fact, for fixed K as $N \to \infty$, the field obtained by rescaling the coarse field onto a box of side length 1 in $\mathbb{R}^2$ converges to a limiting Gaussian field that possesses continuous sample paths on appropriate subsets of $[0,1]^2$ (essentially, away from the boundaries of the sub-boxes).

An important step in our approach is the computation of the tail probabilities of the maximum of the fine field, when restricted to a box of side length N/K, together with the computation of the law of the location of the maximum (in the scale N/K). These computations are done by building on the tail estimates derived in [8], and using a modified second moment method. Another important step is to show that the maximum of the GFF occurs only at points where the fine field is atypically large. Once these two steps are established, we can describe the limit law of the GFF by an appropriate mixture of random variables whose distributions are determined by the tail of the fine field. The mixture coefficients are determined by an (independent) percolation pattern of potential locations of the maximum and by the limiting coarse field.

The structure of the paper is as follows. We first introduce, in Section 2, the coarse and fine fields alluded to above, and restate Theorem 1.1 in terms of this decomposition (see Theorem 2.3). We then introduce, in Section 3, auxiliary processes (branching random walk (BRW) and the modified branching random walk (MBRW) introduced in [5]), and recall several Gaussian tools and estimates that will be used throughout the paper. The long and technical Section 4 is devoted to the derivation of the limiting tail estimates for the maximum of GFF. Once this is established, approximations of the law of η_N^* by local maxima of the fine field are presented in Section 5, which lead, in Section 6, to the proof of Theorem 2.3.

Notation. In order to avoid cumbersome notation throughout the paper, we will assume that $N = 2^n$ for some integer n. The modifications needed to handle general N are minimal and straightforward, and therefore left to the reader. For functions $F(\cdot)$ and $G(\cdot)$, we write $F \lesssim G$ or $F = O(G)$ if there exists an absolute constant $C > 0$ such that $F \leqslant CG$ everywhere in the domain. We write $F \asymp G$ if $F \lesssim G$ and $G \lesssim F$.

2 The coarse and fine fields and the limit result

Let $\eta_{v,N}$ denote the GFF in the box V_N. As mentioned above, we assume for simplicity that $N = 2^n$. Recall that $\eta_N^* = \max_{v \in V_N} \eta_{v,N}$. Fix $K = 2^k$, with k an integer. Divide the box V_N into 4^k non-overlapping shifts of $V_{N/K}$, denoted by $V_N^{K,i}$, and let $\mathcal{F} = \mathcal{F}_{N,K}$ denote the sigma-algebra generated by $\{\eta_{v,N} : v \in \cup_i \partial V_N^{K,i}\}$. (The boundaries of $V_N^{K,i}$ do in fact overlap.) We now define

$$X_v^c = X_{v,N,K}^c = \mathbb{E}(\eta_{v,N} \mid \mathcal{F}_{N,K}), \quad X_v^f = X_{v,N,K}^f = \eta_{v,N} - X_v^c \,. \tag{2}$$

Note that X_v^f vanishes on the boundaries of the boxes $V_N^{K,i}$, that the Gaussian fields $\{X^f\}$ and $\{X^c\}$ are independent, and that the fields $\{X_v^f\}_{v \in V_N^{K,i}}$ are independent for different i and identically distributed as the GFF with zero boundary condition in $V_N^{K,i}$.

Throughout the argument, we will need to consider points that are within an appropriate distance from the boundary of the boxes $V_N^{K,i}$. Toward this end, fix $\delta > 0$ and define the boxes

$$V_N^{K,\delta,i} = \{x \in V_N^{K,i} : d_\infty(x, \partial V_N^{k,i}) \geqslant \delta N/K\}\,; \tag{3}$$

set

$$V_N^{K,\delta} = \cup_i V_N^{K,\delta,i}\,, \quad \Delta_N = \Delta_N^{K,\delta} = V_N \setminus V_N^{K,\delta}\,.$$

Note that $|\Delta_N| \leqslant 4\delta |V_N|$.

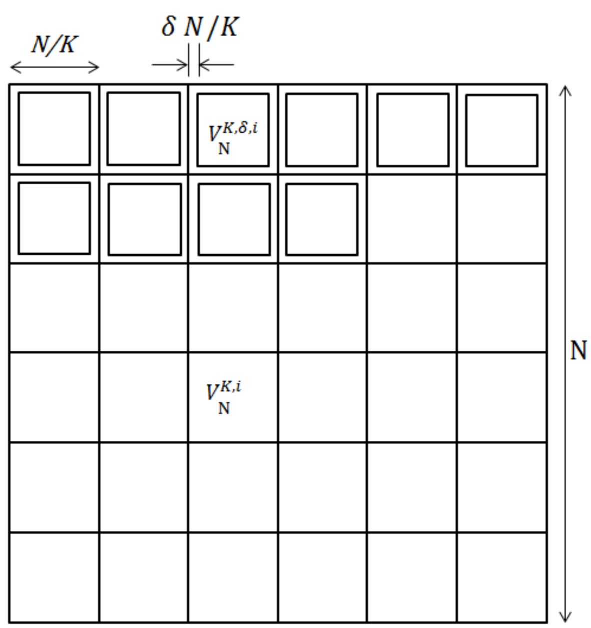

Figure 1: The boxes V_N, $V_N^{K,i}$, $V_N^{K,\delta,i}$

2.1 The coarse field limit

We describe in this short subsection the scaling limit of the Gaussian field $X^c_{\cdot,N,K}$. Introduce the covariance

$$\mathbf{C}^c_{N,K}(v,v') = \mathbb{E}(X^c_{v,N,K} X^c_{v',N,K}) \,, \quad v,v' \in V_N \,. \tag{4}$$

Set $W = [0,1]^2$, and let $W^\delta = W^{K,\delta} = \{x/N : [x] \in V_N^{K,\delta}\}$ be the natural rescaling of $V_N^{K,\delta}$ to W; note that, with $N, K, 1/\delta$ all powers of 2, W^δ does not depend on N. Similarly, define $W^i = W^{K,i} = \{x/N : [x] \in V_N^{K,i}\}$ and $W^{\delta,i} = W^{K,\delta,i} = \{x/N : [x] \in V_N^{K,\delta,i}\}$. Note that $W = \cup_i W^i$ and $W^\delta = \cup_i W^{\delta,i}$.

In what follows, we let $\{w_t\}_{t\geqslant 0}$ denote planar Brownian motion and we write $\mathbb{P}^x(\mathbb{E}^x)$ for probabilities (expectations) involving the path of w_t, with $w_0 = x$. Set $\tau_i = \min\{t : w_t \in \partial W^i\}$ and $\tau = \min\{t : w_t \in \partial W\}$. Introduce the Poisson kernels $p(x,z)$ and $p_i(x,z)$ as the functions satisfying

$$\int_{\partial W} p(x,z) f(z) dz = \mathbb{E}^x(f(w_\tau)) \,, \quad \int_{\partial W^i} p_i(x,z) g(z) dz = \mathbb{E}^x(g(w_{\tau_i})) \,,$$

for any continuous functions f, g. These Poisson kernels give the exit measures of Brownian motion from W and W^i. For $x, x' \in \cup_\delta W^\delta$, set

$$C^c_K(x,x') = \begin{cases} \frac{2}{\pi}\left(\int_{\partial W} p(x,z)\log|z-x'|dz - \int_{\partial W^i} p(x,z)\log|z-x'|dz\right) \,, & x,x' \in W^i \text{ for some } i \\ \frac{2}{\pi}\left(\int_{\partial W} p(x,z)\log\left(\frac{|z-x'|}{|x-x'|}\right)dz\right) \,, & \text{otherwise} \end{cases} \,.$$

Note that, for each fixed δ, C^c_K is uniformly continuous on $W^\delta \times W^\delta$. The following result is crucial for our approach and is easily verified.

Lemma 2.1. *Fix $\delta, K > 0$. Then*

$$|\mathbf{C}^c_{N,K}(v,v') - \mathbf{C}^c_K(v/N, v'/N)| \to_{N\to\infty} 0 \tag{5}$$

uniformly in $(V_N^{K,\delta})^2$.

Note that the limit $\mathbf{C}^c_K$ depends on K and that the convergence rate depends also on δ.

Proof. Employing the orthogonal representation $\eta_{v,N} = X^c_v + X^f_v$,

$$\mathbb{E}(X^c_v \cdot X^c_{v'}) = \mathbb{E}(\eta_{v,N} \cdot \eta_{v',N}) - \mathbb{E}(X^f_v \cdot X^f_{v'}) \,. \tag{6}$$

Recall that

$$\mathbb{E}(\eta_{v,N} \cdot \eta_{v',N}) = \mathbb{E}(\sum_{n=0}^{\tau_N} \mathbf{1}_{\{S_n = v'\}}) \,,$$

where $\{S_n\}$ denotes two-dimensional simple random walk starting from v and τ_N is the first exit time from V_N (see [5] or [8]). Consider the potential kernel for two-dimensional simple random walk,

$$a(x) = \frac{2}{\pi}\log|x| + \frac{2\bar{\gamma} + \log 8}{\pi} + O(|x|^{-2}) \,, \tag{7}$$

with $a(0) = 0$, and where $\bar{\gamma}$ is the Euler constant (see [12, Theorem 4.4.4]). From [12, Theorem 4.6.2],

$$\mathbb{E}(\eta_{v,N} \cdot \eta_{v',N}) = \mathbb{E}(\sum_{n=0}^{\tau_N} \mathbf{1}_{\{S_n = v'\}}) = \sum_{z\in\partial V_N} \mathbb{P}^v(S_{\tau_N} = z) a(z - v') - a(v' - v) \,. \tag{8}$$

Similarly, when $v, v' \in V_N^{K,i}$ for some i,

$$\mathbb{E}(X_v^f \cdot X_{v'}^f) = \sum_{z \in \partial V_N^{K,i}} \mathbb{P}^v(S_{\tau_N^{(i)}} = z) a(z - v') - a(v' - v) , \tag{9}$$

where τ_N^i denotes the first exit time from $V_N^{K,i}$. On the other hand, when v, v' do not belong to the same box $V_N^{K,i}$, $\mathbb{E}(X_v^f \cdot X_{v'}^f) = 0$. The conclusion follows from (4), the convergence of simple random walk to Brownian motion, (7), (8) and (9). $\square$

It follows from Lemma 2.1 that $\mathbf{C}_K^c$ is a covariance function and therefore there exists a mean zero Gaussian field $\{Z_{K,\delta}^c(x)\}_{x \in W^{K,\delta}}$ with covariance $\mathbf{C}_K^c$.

2.2 The fine field limit

We fix a function $g(K)$ that grows to ∞ with K. (The choice $g(K) = \alpha \log \log K$, with an appropriately chosen $\alpha > 0$, will be used in Proposition 5.2.) The following result will be demonstrated in Section 4.

Proposition 2.2. *Define the event*

$$\mathcal{A}_{N,K} = \{ \max_{v \in V_{N/K}} \eta_{v,N/K} \geqslant m_{N/K} + g(K) \} .$$

There exists an absolute constant $\alpha^* > 0$ so that

$$\lim_{K \to \infty} \lim_{N \to \infty} \frac{\mathrm{e}^{\sqrt{2\pi} g(K)}}{g(K)} \mathbb{P}(\mathcal{A}_{N,K}) = \alpha^* . \tag{10}$$

Choose $v^* = v^*_{N/K}$ so that $\max_{v \in V_{N/K}} \eta_{v,N/K} = \eta_{v^*,N/K}$. There exists a continuous function $\psi : (0,1)^2 \to (0, \infty)$, with $\int_{[0,1]^2} \psi(y) dy = 1$, such that, for any open set $A \subset (0,1)^2$ and any sequence $x_K \geqslant 0$ not depending on N,

$$\lim_{K \to \infty} \lim_{N \to \infty} \frac{\mathrm{e}^{\sqrt{2\pi} x_K} g(K)}{g(K) + x_K} \mathbb{P}(\max_{v \in V_{N/K}} \eta_{v,N/K} \geqslant m_{N/K} + g(K) + x_K, K v^*/N \in A | \mathcal{A}_{N,K}) = \int_A \psi(y) dy , \tag{11}$$

with convergence being uniform in the sequence x_K.

2.3 The limit process

Fix K, δ. Consider the unit square $[0,1]^2$ and partition it into non-overlaping squares $\{\mathcal{W}_i\}_{i=1,\dots,K^2}$ of side length $1/K$ each. (We omit K from the notation.) Define $\mathcal{W}_i^\delta$ to be the subset of $\mathcal{W}_i$ consisting of points whose distance to the boundary of $\mathcal{W}_i$ is at least δ/K.

Let ψ and α^* be as in Proposition 2.2. In each $\mathcal{W}_i$, choose a point z_i^K that is distributed according to the scaled analog of ψ, with z_i^K being chosen independently for different i. Let $\{\wp_i^K\}_{i=1,\dots,K^2}$ denote independent Bernoulli random variables with $\mathbb{P}(\wp_i^K = 1) = \alpha^* g(K) \mathrm{e}^{-\sqrt{2\pi} g(K)}$, and let $\{Y_i^K\}_{i=1,\dots,K^2}$ denote independent random variables satisfying

$$\mathbb{P}(Y_i^K \geqslant x) = \frac{g(K) + x}{g(K)} \mathrm{e}^{-\sqrt{2\pi} x} , \quad x \geqslant 0 . \tag{12}$$

Recall the limiting coarse field $Z^c_{K,\delta}$ and define

$$G_i^{K,\delta} = 1_{\{z_i^K \in \mathcal{W}_i^\delta\}} \wp_i^K (Y_i^K + g(K)) + Z^c_{K,\delta}(z_i^K).$$

Set $G^*_{K,\delta} = \max_i G_i^{K,\delta}$ and denote by $\mu_{K,\delta}$ its law. Note that $\mu_{K,\delta}$ does not depend on N.

Let $d(\cdot,\cdot)$ denote a metric that is compatible with the weak convergence of probability measures on $\mathbb{R}$. Our main result in the paper is the following.

Theorem 2.3. *Let μ_N denote the law of $\max_{v\in V_N} \eta_v^N - m_N$. Then, with $\mu_{K,\delta}$ defined as above,*

$$\limsup_{\delta\searrow 0} \limsup_{K\to\infty} \limsup_{N\to\infty} d(\mu_N, \mu_{K,\delta}) = 0\,. \tag{13}$$

In particular, there exists a probability measure μ_∞ on $\mathbb{R}$ such that $d(\mu_N, \mu_\infty) \to_{N\to\infty} 0$.

3 Preliminaries

3.1 Branching random walk and modified branching random walk

We first briefly review the construction of branching random walk (BRW) and modified branching random walk (MBRW), which we construct so as to simplify comparisons with the GFF. As before, we let V_N denote the box of side length N placed in $\mathbb{Z}^2$ such that the lower left corner is at the origin, and set $n = \log_2 N$. For $j \in [0,\dots,n]$, let $\mathfrak{B}_j$ be the collection of squared boxes in $\mathbb{Z}^2$ of side length 2^j with corners in $\mathbb{Z}^2$, and let $\mathfrak{BD}_j$ denote the subset of $\mathfrak{B}_j$ consisting of squares of the form $([0,2^j-1]\cap\mathbb{Z})^2 + (i_1 2^j, i_2 2^j)$, with $i_1, i_2 \in \mathbb{N}$. For $v \in \mathbb{Z}^2$, let $\mathfrak{B}_j(v) = \{B \in \mathfrak{B}_j : v \in B\}$ be the collection of boxes in $\mathfrak{B}_j$ that contain v, and define $\mathfrak{BD}_j(v)$ to be the (unique) box in $\mathfrak{BD}_j$ that contains v. Furthermore, denote by $\mathfrak{B}_{N,j}$ the subset of $\mathfrak{B}_j$ consisting of boxes whose lower left corners are in V_N. Let $\{\bar\phi_{N,j,B}\}_{j\geqslant 0, B\in\mathcal{BD}_j}$ be i.i.d. Gaussian variables with variance $\frac{2\log 2}{\pi}$, and define a branching random walk

$$\vartheta_{v,N} = \textstyle\sum_{j=0}^n \bar\phi_{N,j,\mathcal{BD}_j(v)}\,. \tag{14}$$

For $j \in [0,\dots,n]$ and $B \in \mathfrak{B}_{N,j}$, let $\phi_{N,j,B}$ be independent centered Gaussian variables with $\mathrm{Var}(\phi_{N,j,B}) = \frac{2\log 2}{\pi}\cdot 2^{-2j}$, and define

$$\phi_{N,j,B} = \phi_{N,j,B'}, \text{ for } B \sim_N B' \in \mathcal{B}_{B,j}\,, \tag{15}$$

where $B \sim_N B'$ if and only if there exist $i_1, i_2 \in \mathbb{Z}$ such that $B = (i_1 N, i_2 N) + B'$. (Note that, for any $B \in \mathcal{B}_j$, there exists a unique $B' \in \mathfrak{B}_{N,j}$ such that $B \sim_N B'$.) Let $d_N(u,v) = \min_{w\sim_N v}|u-w|$ be the ℓ^2 distance between u and v when considering V_N as a torus, for all $u, v \in V_N$. Finally, we define the MBRW $\{\xi_{v,N} : v \in V_N\}$ by

$$\xi_{v,N} = \textstyle\sum_{j=0}^n \sum_{B\in\mathfrak{B}_j(v)} \phi_{N,j,B}\,. \tag{16}$$

The motivation for introducing MBRW is that the MBRW approximates the GFF with high precision. That is, the covariance structure of the MBRW approximates that of the GFF well. This is elaborated in the next lemma.

Lemma 3.1. *For any $0 < \delta < 1/100$, there exists a constant $C = C_\delta$ such that, for all n,*

$$|\mathrm{Cov}(\xi_{u,N}, \xi_{v,N}) - \tfrac{2\log 2}{\pi}(n - \log_2(d_N(u,v)))| \leqslant C, \text{ for all } u, v \in V_N\,,$$
$$\left|\mathrm{Cov}(\eta_{u,N}, \eta_{v,N}) - \tfrac{2\log 2}{\pi}(n - (0 \vee \log_2|u-v|))\right| \leqslant C, \text{ for all } u, v \in V_N'\,,$$

where V_N' is a box of side length $(1-2\delta)N$ placed at the center of V_N.

The preceding lemma is slightly stronger than [5, Lemma 2.2], but the steps in the proof of [5] remain essentially the same.

3.2 A few Gaussian inequalities

We will need the following two Gaussian comparison inequalities: the Sudakov-Fernique inequality, which compares the expected maximum of Gaussian processes (see, e.g., [10] for a proof), and Slepian's comparison lemma [14], which compares the maximum of Gaussian processes in the sense of "stochastic domination".

Lemma 3.2 (Sudakov-Fernique)**.** *Let $\mathcal{A}$ be an arbitrary finite index set and let $\{X_a\}_{a\in\mathcal{A}}$ and $\{Y_a\}_{a\in\mathcal{A}}$ be two centered Gaussian processes such that*

$$\mathbb{E}(X_a - X_b)^2 \geqslant \mathbb{E}(Y_a - Y_b)^2\,, \text{ for all } a, b \in \mathcal{A}\,. \tag{17}$$

Then $\mathbb{E}\max_{a\in\mathcal{A}} X_a \geqslant \mathbb{E}\max_{a\in\mathcal{A}} Y_a$.

Lemma 3.3 (Slepian)**.** *Let $\mathcal{A}$ be an arbitrary finite index set and let $\{X_a\}_{a\in\mathcal{A}}$ and $\{Y_a\}_{a\in\mathcal{A}}$ be two centered Gaussian processes such that* (17) *holds and* $\mathrm{Var}\, X_a = \mathrm{Var}\, Y_a$ *for all $a \in \mathcal{A}$. Then $\mathbb{P}(\max_{a\in\mathcal{A}} X_a \geqslant \lambda) \geqslant \mathbb{P}(\max_{a\in\mathcal{A}} Y_a \geqslant \lambda)$, for all $\lambda \in \mathbb{R}$.*

The following Borell-Tsirelson inequality is a central result in the theory of concentration of measure; see, for example, [13, Theorem 7.1, Equation (7.4)].

Lemma 3.4. *Consider a Gaussian process $\{\eta_x : x \in V\}$ and set $\sigma = \sup_{x\in V}(\mathbb{E}(\eta_x^2))^{1/2}$. Then, for $\alpha > 0$,*

$$\mathbb{P}\left(\left|\sup_{x\in V}\eta_x - \mathbb{E}\sup_{x\in V}\eta_x\right| > \alpha\right) \leqslant 2\exp(-\alpha^2/2\sigma^2)\,.$$

We will often need to control the expectation of the maximum of a Gaussian field in terms of its covariance structure. This is achieved by Fernique's criterion [10]. We quote a version suited to our needs, which follows straightforwardly from the version in [2, Theorem 4.1] by using, as the majorizing measure, the normalized counting measure on B.

Lemma 3.5. *There exists a universal constant $C_F > 0$ with the following property. Let $B \subset \mathbb{Z}^2$ denote a (discrete) box of side length b and assume $\{G_v\}_{v\in B}$ is a mean zero Gaussian field satisfying*

$$\mathbb{E}(G_v - G_u)^2 \leqslant |u - v|/b\,, \text{ for all } u, v \in B\,.$$

Then

$$\mathbb{E}\max_{v\in B} G_v \leqslant C_F\,. \tag{18}$$

3.3 A Brownian motion estimate

In this subsection, we show that the probabilities for Brownian motion to stay below two close curves are asymptotically the same.

Lemma 3.6. *Let $C > 0$ be a fixed absolute constant, and let $\{W_s : s \geqslant 0\}$ be a mean zero Brownian motion, started at 0, with a fixed variance rate σ^2. For $y > 1$ and $t > 0$, define densities $\mu_{t,y}(\cdot)$ and $\mu^*_{t,y}(\cdot)$ such that, for all $I \subset \mathbb{R}$,*

$$\begin{aligned} &\mathbb{P}(W_t \in I; W_s \leqslant y \text{ for all } 0 \leqslant s \leqslant t) = \int_I \mu_{t,y}(x)dx\,, \\ &\mathbb{P}(W_t \in I; W_s \leqslant y + y^{1/20} + C(s \wedge (t-s))^{1/20} \text{ for all } 0 \leqslant s \leqslant t) = \int_I \mu^*_{t,y}(x)dx\,. \end{aligned} \tag{19}$$

Then there exists δ_y, with $\delta_y \searrow_{y\to\infty} 0$, such that, for all $x \leqslant 0$ and $t > 0$,

$$\mu^*_{t,y}(x) \leqslant (1+\delta_y)\mu_{t,y}(x)\,. \tag{20}$$

For all $x \leqslant y + y^{1/20}$,

$$\mu^*_{t,y}(x) \lesssim y(y + y^{1/20} - x)n^{-3/2}\,. \tag{21}$$

Furthermore,

$$\frac{\mu^*_{t,y}(x_1)}{\mu^*_{t,y}(x_2)} \leqslant \mathrm{e}^{-\frac{x_1^2 - x_2^2}{2t\sigma^2}} \text{ for all } 0 \leqslant x_2 \leqslant x_1 \leqslant y + y^{1/20}\,. \tag{22}$$

Proof. In order to show (22), note that $\{W_s - sW_t/t : 0 \leqslant s \leqslant t\}$ is distributed as a Brownian bridge that is independent of the value W_t. So, the probability for the Brownian bridge $\{W_s - sW_t/t : 0 \leqslant s \leqslant t\}$ to stay below a given curve, after conditioning on $W_t = x$, is decreasing with x. Therefore, the ratio between $\mu^*_{t,y}(x_1)$ and $\mu^*_{t,y}(x_2)$ is bounded above by the ratio of the densities for the Brownian motion at x_1 and x_2.

We next prove (20). By the reflection principle, for all $x \geqslant 0$,

$$\mu_{t,y}(y - x) = \frac{1}{\sqrt{2\pi t}\sigma}(\mathrm{e}^{-\frac{(x-y)^2}{2t\sigma^2}} - \mathrm{e}^{-\frac{(x+y)^2}{2t\sigma^2}}). \tag{23}$$

Let τ be a global maximizer of $\{W_s : 0 \leqslant s \leqslant t\}$, i.e., $W_\tau = \max_{0\leqslant s\leqslant t} W_s$, and write $\psi_{t,y}(s) = y + y^{1/20} + C(s \wedge (t-s))^{1/20}$ and $\psi^*_{t,y}(j) = \max_{s\in[j,j+1]} \psi_{t,y}(s)$. Summing over different j for $\tau \in [j, j+1]$ and integrating over locations for W_j, we obtain from the Markov property applied at time j and then again at time τ that, for all $x \leqslant 0$,

$$\begin{aligned} &\mu^*_{t,y}(x) - \mu_{t,y}(x) \\ &\leqslant \sum_{j=0}^{\lfloor t\rfloor} \int_{-\infty}^{\psi^*_{t,y}(j)} \mu_{j,\psi^*_{t,y}(j)}(\lambda)\mathbb{P}(\lambda + \max_{0\leqslant s\leqslant 1} W_s \geqslant y) \max_{j-1\leqslant s\leqslant j} \max_{y\leqslant y'\leqslant \psi^*_{t,y}} \mu_{t-s,\psi^*_{t,y}(j)-y'}(x - y')d\lambda\,. \end{aligned}$$

(Note that, on the relevant event, $W_\tau \leqslant \psi_{t,y}(\tau)$.) By (22), $\max_{y\leqslant y'\leqslant \psi^*_{t,y}} \mu_{t-s,\psi^*_{t,y}(j)-y'}(x - y') \leqslant \mu_{t-s,\psi^*_{t,y}(j)-y}(x - y)$. Therefore,

$$\mu^*_{t,y}(x) - \mu_{t,y}(x) \lesssim \sum_{j=0}^{\lfloor t\rfloor} \int_{-\infty}^{\psi^*_{t,y}(j)} \mu_{j,\psi^*_{t,y}(j)}(\lambda)\mathrm{e}^{-\frac{(\lambda-y)_-^2}{2\sigma^2}} \max_{j-1\leqslant s\leqslant j} \mu_{t-s,\psi^*_{t,y}(j)-y}(x-y)d\lambda =: \sum_{j=0}^{\lfloor t\rfloor} \Phi_j\,, \tag{24}$$

where $(\lambda - y)_- = \mathbf{1}_{\lambda \leqslant y}|\lambda - y|$. Applying (23), we obtain that, for both $j \leqslant y^2/(\log y)^2$ and $j \geqslant t - y^2/(\log y)^2$,

$$\Phi_j \lesssim \mathrm{e}^{-(\log y)^2/4\sigma^2}(y-x)t^{-3/2}\,. \tag{25}$$

Again, by the reflection principle, for all $\lambda \leqslant \psi^*_{t,y}(j)$, $x \leqslant 0$ and $j-1 \leqslant s \leqslant j$,

$$\mu_{j,\psi^*_{t,y}(j)}(\lambda) \lesssim j^{-3/2}(\psi^*_{t,y}(j) - \lambda)\psi^*_{t,y}(j)$$
$$\mu_{t-s,\psi^*_{t,y}(j)-y}(x-y) \lesssim (\psi^*_{t,y}(j) - y)(\psi^*_{t,y}(j) - x)(t-j)^{-3/2}\,.$$

For $y^2/(\log y)^2 \leqslant j \leqslant t - y^2/(\log y)^2$, plugging the above estimates into the integral in (24) leads to

$$\Phi_j \lesssim (\psi^*_{t,y}(j) - y)^3 j^{-3/2}(t-j)^{-3/2}(\psi^*_{t,y}(j) - x)\psi^*_{t,y}(j)\,. \tag{26}$$

Plugging (25) and (26) into (24) and summing over j, we obtain

$$\mu^*_{t,y}(x) - \mu_{t,y}(x) \lesssim (y-x)yt^{-3/2}(y^2\mathrm{e}^{-\frac{(\log y)^2}{4\sigma^2}} + y^{-1/2}\log y) \lesssim \mu_{t,y}(x)(y^2\mathrm{e}^{-\frac{(\log y)^2}{4\sigma^2}} + y^{-1/2}\log y)\,,$$

which completes the proof of (20) (where we have used $\mu_{t,y}(x) \asymp y(y-x)t^{-3/2}$).

It remains to show (21). The result follows by using the same decomposition for the location of the global maximizer τ of Brownian motion in a manner analogous to the proof of (20), together with similar straightforward computations. We omit further details. $\square$

3.4 Refined estimates on the right tail of the maximum for the GFF

In [8], it is shown that

$$\mathbb{P}(\eta^*_N > m_N + \lambda) \asymp \lambda \mathrm{e}^{-\sqrt{2\pi}\lambda} \text{ for all } 1 \leqslant \lambda \leqslant \sqrt{\log N}\,. \tag{27}$$

In this subsection, we give a preliminary upper bound on the right tail of the maximum of GFF over subsets of V_N, for values of λ that include $\lambda \gg \sqrt{\log N}$. To do this, we first obtain an upper bound on the probability of BRW taking atypically large values. We will use the notation of Subsection 3.1 and, for convenience, will view each $\vartheta_{v,N}$ as the value at time n of a Brownian motion $\{\vartheta_{v,N}(t) : 0 \leqslant t \leqslant n\}$ with variance rate $\frac{2\log 2}{\pi}$. More precisely, we associate to each Gaussian variable $\bar\phi_{N,j,B}$ an independent Brownian motion with variance rate $\frac{2\log 2}{\pi}$ that runs for one unit of time and ends at the value $\bar\phi_{N,j,B}$. For $\beta > 0$, define

$$G_N(\beta) = \bigcup_{v \in V_N} \bigcup_{0 \leqslant t \leqslant n} \{\vartheta_{v,N}(t) \geqslant \beta + 1 + \frac{m_N}{n}t + 10(\log(t \wedge (n-t)))_+\}\,. \tag{28}$$

Lemma 3.7. *There exists an absolute constant $C > 0$ such that $\mathbb{P}(G_N(\beta)) \lesssim \beta \mathrm{e}^{-\sqrt{2\pi}\beta}\mathrm{e}^{-\beta^2/Cn}$ for all $\beta > 0$.*

Proof. We may assume that $\beta > \beta_0$, for some $\beta_0 > 0$ large. For any $v \in V_N$, write $\bar\vartheta_{v,N}(t) = \vartheta_{v,N}(t) - \frac{m_N t}{n}$. Define the probability measure $\mathbb{Q}$ by

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = \mathrm{e}^{-\frac{\pi m_N}{(2\log 2)n}\bar\vartheta_{v,N}(n) - \frac{\pi m_N^2}{(4\log 2)n}}\,. \tag{29}$$

Then, under $\mathbb{Q}$, $\{\bar{\vartheta}_{v,N}(t) : 0 \leqslant t \leqslant n\}$ is a mean zero Brownian motion with variance rate $\frac{2\log 2}{\pi}$. For $0 \leqslant t \leqslant n$, write $\psi_{N,t,\beta} = \beta + 1 + 10(\log(t \wedge (n-t)))_+$. For $j \in [0, \ldots, n]$, let $\chi_{N,j}(\cdot)$ be the density function such that, for all $I \subseteq \mathbb{R}$,

$$\mathbb{P}(\bar{\vartheta}_{v,N}(t) \leqslant \psi_{N,t,\beta} \text{ for all } t \leqslant j, \bar{\vartheta}_{v,N}(j) \in I) = \int_I \chi_{N,j}(x)dx\,.$$

By a straightforward union bound,

$$\mathbb{P}(G_N(\beta)) \leqslant \sum_{j=1}^{n-1} 4^j \int_{-\infty}^{\psi_{N,j,\beta}} \chi_{N,j}(x)\mathbb{P}(\exists s \in [j, j+1] : x + \bar{\vartheta}_{v,N}(s) - \bar{\vartheta}_{v,N}(j) \geqslant \psi_{N,s,\beta})dx\,. \tag{30}$$

Applying (21) (with $t = j$ and $y + y^{1/20} = \psi_{N,j,\beta}$), we see that, for $x \leqslant \psi_{N,j,\beta}$,

$$(\frac{d\mathbb{P}}{d\mathbb{Q}}(x))^{-1}\chi_{N,j}(x) \lesssim j^{-3/2}\psi_{N,j,\beta}(\psi_{N,j,\beta} - x)\,.$$

Combined with (29), this yields

$$\chi_{N,j}(x) \lesssim 4^{-j}\mathrm{e}^{-(\sqrt{2\pi}-O((\log n)/n))x}\mathrm{e}^{-\pi x^2/4j\log 2}\psi_{N,j,\beta}(\psi_{N,j,\beta} - x)\,. \tag{31}$$

Note that $\max_{0\leqslant s\leqslant 1}\vartheta_{v,N}(s)$ has the same distribution as $|Z|$, for $Z \sim N(0,\gamma^2)$ and $\gamma = \sqrt{2\log 2/\pi}$. Therefore,

$$\mathbb{P}(\exists s \in [j, j+1] : x + \bar{\vartheta}_{v,N}(s) - \bar{\vartheta}_{v,N}(j) \geqslant \psi_{N,s,\beta}) \lesssim \exp(-(\min_{s\in[j,j+1]} \psi_{N,s,\beta} - x)^2/(2\gamma^2))\,. \tag{32}$$

Since the right hand side of (32) decays (essentially) exponentially in $|\psi_{N,j,\beta} - x|^2$, the typical value of x that will contribute to the integral of (30) will be close to $\psi_{N,j,\beta}$. With this observation in mind, plugging (32) and (31) into (30), we obtain that, for an absolute constant $C > 0$,

$$\mathbb{P}(G_N(\beta)) \lesssim \sum_{j=1}^{n} 4^j 4^{-j}(j \wedge (n+1-j))^{-2}\mathrm{e}^{-\sqrt{2\pi}\beta}\mathrm{e}^{-\beta^2/Cn} \lesssim \beta\mathrm{e}^{-\sqrt{2\pi}\beta}\mathrm{e}^{-\beta^2/Cn}\,,$$

where the power 2 in $(j \wedge (n+1-j))^2$ is obtained from the slackness term $10(\log(j \wedge (n-j)))_+$ (with room to spare). $\square$

Lemma 3.8. *There exists an absolute constant $C > 0$ such that, for all $N \in \mathbb{N}$ and $z \geqslant 1$,*

$$\mathbb{P}(\max_{v\in V_N} \eta_{v,N} \geqslant m_N + z) \lesssim z\mathrm{e}^{-\sqrt{2\pi}z}\mathrm{e}^{-C^{-1}z^2/n}\,. \tag{33}$$

Furthermore,

$$\mathbb{P}(\max_{v\in A} \eta_{v,N} \geqslant m_N + z - y) \lesssim (\frac{|A|}{|V_N|})^{1/2} z\mathrm{e}^{-\sqrt{2\pi}(z-y)} \text{ for all } z \geqslant 1, y \geqslant 0\,, \tag{34}$$

for all $A \subseteq V_N$.

Proof. By the same argument as in [8, Lemma 2.6], for an absolute constant $\kappa \geqslant 0$ and any $A \subseteq V_N$,

$$\mathbb{P}(\max_{v\in A} \eta_{v,N} \geqslant \lambda) \leqslant \mathbb{P}(\max_{v\in 2^\kappa A} \vartheta_{v,2^\kappa N} \geqslant \lambda) \text{ for all } \lambda \in \mathbb{R}\,,$$

where $\{\vartheta_{v,2^\kappa N}\}$ is a BRW on 2D box of side length $2^\kappa N$. By this inequality and a change of variable (replacing $2^\kappa N$ by N), it suffices to prove (33) and (34) for BRW. For $\beta > 0$, define

$$F_{v,N}(\beta, z) = \{\vartheta_{v,N}(t) \leqslant \beta + 1 + \frac{m_N}{n} t + 10\log(t \wedge (n-t))_+ \text{ for all } 0 \leqslant t \leqslant n; \vartheta_{v,N} \geqslant m_N + z\}\,.$$

We first prove the BRW version for (33), using the notation $\mu_{n,z}(\cdot)$ and $\mu^*_{n,z}(\cdot)$ from (19) (with variance rate $\sigma^2 = \frac{2\log 2}{\pi}$). By (21), and using $d\mathbb{P}/d\mathbb{Q}$ as in Lemma 3.7 (with z' satisfying $z' + (z')^{1/20} = z+1$),

$$\begin{aligned}\mathbb{P}(F_{v,N}(z,z)) &= \int_z^{z+1} \frac{d\mathbb{P}}{d\mathbb{Q}}(x+m_N)\mu^*_{n,z'}(x)dx \\ &\leqslant \int_z^{z+1} 4^{-n} n^{3/2} \mathrm{e}^{-(\sqrt{2\pi} - O((\log n)/n))x} \mathrm{e}^{-\pi z^2/4n\log 2} \cdot z n^{-3/2} dx \\ &\lesssim 4^{-n} z \mathrm{e}^{-\sqrt{2\pi}z} \mathrm{e}^{-\pi z^2/4n\log 2} \mathrm{e}^{O((\log n)/n)z}\,. \end{aligned} \tag{35}$$

Inequality (33) follows by summing the above inequality over $v \in V_N$ and applying Lemma 3.7.

We next demonstrate (34). First note that, if $z - y + (|V_N|/|A|)^{1/4} \leqslant 1$, then (34) holds automatically. We next consider the case when $z - y + (|V_N|/|A|)^{1/4} \geqslant 1$ and set $\beta = z - y + (|V_N|/|A|)^{1/4}$. By Lemma 3.7,

$$\mathbb{P}(G_N(\beta)) \lesssim \mathrm{e}^{-\sqrt{2\pi}(z-y)} \mathrm{e}^{-\sqrt{2\pi}(|V_N|/|A|)^{1/4}}\,. \tag{36}$$

Analogous to the derivation of (35), we obtain

$$\mathbb{P}(F_{v,N}(\beta, z-y)) \lesssim 4^{-n}(z - y + (|V_N|/|A|)^{1/4})(|V_N|/|A|)^{1/4} \mathrm{e}^{-\sqrt{2\pi}(z-y)}\,.$$

Summation over $v \in A$ and application of (36) implies (34). □

3.5 Robustness of the maximum for the GFF

The following lemma shows a form of robustness for the maximum of the GFF under perturbation.

Lemma 3.9. *Let $\phi_{u,N}$ be independent variables such that, for all $u \in V_N$,*

$$\mathbb{P}(\phi_{u,N} \geqslant 1 + y) \leqslant \mathrm{e}^{-y^2}\,.$$

There exists an absolute constant $C > 0$ such that, for any $\varepsilon > 0$, $N \in \mathbb{N}$ and $x \geqslant -\varepsilon^{-1/2}$,

$$\mathbb{P}(\max_{u\in V_N}(\eta_{u,N} + \varepsilon\phi_{u,N}) \geqslant m_N + x) \leqslant \mathbb{P}(\max_{u\in V_N} \eta_{u,N} \geqslant m_N + x - \sqrt{\varepsilon})(1 + O(\mathrm{e}^{-C^{-1}\varepsilon^{-1}}))\,. \tag{37}$$

Furthermore, $\mathbb{E}\max_{u\in V_N}(\eta_{u,N} + \varepsilon\phi_{u,N}) \leqslant \mathbb{E}\max_{u\in V_N} \eta_{u,N} + C\sqrt{\varepsilon} + C\varepsilon^2 \mathbf{1}_{\varepsilon\geqslant 1}$.

Proof. The case for $\varepsilon > 1$ is simpler and follows from similar arguments to those when $\varepsilon \leqslant 1$. We therefore omit the proof for the case $\varepsilon > 1$. Setting $\Gamma_y = \{u \in V_N : y/2 \leqslant \varepsilon\phi_{u,N} \leqslant y\}$, we have

$$\mathbb{P}(\max_{u\in V_N}(\eta_{u,N} + \varepsilon\phi_{u,N}) \geqslant m_N + x) \leqslant \mathbb{P}(\max_{u\in V_N} \eta_{u,N} \geqslant m_N + x - \sqrt{\varepsilon}) + \sum_{i=0}^{\infty} \mathbb{E}(\mathbb{P}(\max_{u\in\Gamma_{2^i\sqrt{\varepsilon}}} \eta_{u,N} \geqslant m_N + x - 2^i\sqrt{\varepsilon} \mid \Gamma_{2^i\sqrt{\varepsilon}})), \quad (38)$$

where the conditioning on the right side of the last display means conditioning on the locations of points $u \in V_N$ where the random field $\phi_{u,N}$ takes the prescribed values. By Lemma 3.8,

$$\sum_{i=0}^{\infty} \mathbb{E}(\mathbb{P}(\max_{u\in\Gamma_{2^i\sqrt{\varepsilon}}} \eta_{u,N} \geqslant m_N + x - 2^i\sqrt{\varepsilon} \mid \Gamma_{2^i\sqrt{\varepsilon}})) \lesssim \frac{x \vee 1}{\mathrm{e}^{\sqrt{2\pi}x}} \sum_{i=0}^{\infty} \mathbb{E}(|\Gamma_{2^i\sqrt{\varepsilon}}|/N^2)^{1/2}\mathrm{e}^{\sqrt{2\pi}2^i\sqrt{\varepsilon}}dt\,.$$

A simple computation yields $\mathbb{E}|\Gamma_{2^i\sqrt{\varepsilon}}/N^2|^{1/2} \leqslant \mathrm{e}^{-4^i(C\varepsilon)^{-1}}$, for an absolute constant $C > 0$. Combining this with the last two displays, it follows that, for an absolute constant $C^* > 0$,

$$\sum_{i=0}^{\infty} \mathbb{E}(\mathbb{P}(\max_{u\in\Gamma_{2^i\sqrt{\varepsilon}}} \eta_{u,N} \geqslant m_N + x - 2^i\sqrt{\varepsilon} \mid \Gamma_{2^i\sqrt{\varepsilon}})) \lesssim \frac{x \vee 1}{\mathrm{e}^{\sqrt{2\pi}x}}\mathrm{e}^{-(C^*\varepsilon)^{-1}}\,.$$

Together with (27) and (38), this implies (37).

We next estimate the expectation of the maximum. Let $\bar{M}_N = \eta_N^* \vee (m_N - \varepsilon^{-1/2})$. By the estimate on the left tail in [7, Theorem 1.1], $\mathbb{E}\bar{M}_N - \eta_N^* \leqslant C\sqrt{\varepsilon}$, where C is an absolute constant. Set $\tilde{M}_N = \max_{u\in V_N} \eta_{u,N} + \varepsilon\phi_{u,N}$. By (37),

$$\mathbb{E}(\tilde{M}_N - \bar{M}_N) \lesssim \sqrt{\varepsilon} + \mathrm{e}^{-C^{-1}\varepsilon^{-1}} \int_{-\varepsilon^{-1/2}}^{\infty} \mathbb{P}(\max_{u\in\tilde{V}_N} \eta_{u,N} \geqslant m_N + x - \sqrt{\varepsilon})dx \lesssim \sqrt{\varepsilon}\,,$$

which completes the proof of the lemma. □

3.6 A covariance computation

We will also need the following estimate on the coarse field $X^c_{\cdot,N,K}$.

Lemma 3.10. *There exists a constant c_δ, not depending on K, N, such that, for all i and all $v, v' \in V_N^{K,\delta,i}$,*

$$\mathbb{E}(X^c_v - X^c_{v'})^2 \leqslant c_\delta \frac{|v - v'|}{N/K}\,. \quad (39)$$

Proof. We assume that i is fixed and suppress it from the notation, since the estimates will not depend on i. For convenience, set $V_N^{K,i} = V_{N/K}$. Recalling (7) and (8), we obtain

$$|\mathbb{E}(\eta_{v,N} - \eta_{v',N})^2 - 2a(v - v')| \leqslant 2 \max_{z\in\partial V_N} |a(z - v) - a(z - v')| \leqslant 4\frac{|v - v'|}{\delta N/K} + CK^2/\delta^2N^2\mathbf{1}_{v\neq v'}.$$

(We have used the lower bound $\delta N/K$ on the distances from v and v' to ∂V_N.) Applying the same estimate again, this time to $V_{N/K}$ and using (6) and (9), (39) follows. □

4 The limiting tail of the GFF maximum

Recall that $\{\eta_{v,N} : v \in V_N\}$ denotes the GFF on the two-dimensional box V_N and that $\eta_N^* = \max_{v\in V_N} \eta_{v,N}$. For an open set $A \subseteq (0,1)^2$, let $NA = \{v \in V_N : v/N \in A\}$. The main result of this section is the following proposition.

Proposition 4.1. *There exists an absolute constant $\alpha^* > 0$ such that*

$$\lim_{z\to\infty} \limsup_{N\to\infty} |z^{-1} \mathrm{e}^{\sqrt{2\pi} z} \mathbb{P}(\eta_N^* \geqslant m_N + z) - \alpha^*| = 0\,.$$

Furthermore, there exists a continuous function $\psi : (0,1)^2 \mapsto (0,\infty)$ with $\int_{[0,1]^2} \psi(x)dx = 1$ such that, for any open set $A \subseteq (0,1)^2$,

$$\lim_{z\to\infty} \limsup_{N\to\infty} |z^{-1} \mathrm{e}^{\sqrt{2\pi} z} \mathbb{P}(\max_{v\in NA} \eta_{v,N} \geqslant m_N + z) - \alpha^* \int_A \psi(x)dx| = 0\,.$$

The following corollary follows quickly from Proposition 4.1.

Corollary 4.2. *For any open box $A \subseteq (0,1)^2$,*

$$\lim_{z\to\infty} \limsup_{N\to\infty} z^{-1} \mathrm{e}^{\sqrt{2\pi} z} \mathbb{P}(\max_{v\in NA} \eta_{v,N} \geqslant m_N + z, \max_{v\in V_N\setminus NA} \eta_{v,N} \geqslant m_N + z) = 0\,.$$

Proof. For $0 < \delta < 1/10$, let $V_\delta = \{v \in [0,1]^2 : \mathrm{dist}(v, \overline{A}) \leqslant \delta\} \cup ([0,1]^2 \setminus (\delta, 1-\delta)^2)$, where $\overline{A}$ denotes the closure of A. Consider the open sets A and V_δ^c. By Proposition 4.1 and the inclusion-exclusion principle,

$$\lim_{z\to\infty} \limsup_{N\to\infty} z^{-1} \mathrm{e}^{\sqrt{2\pi} z} \mathbb{P}(\max_{v\in NA} \eta_{v,N} \geqslant m_N + z, \max_{v\in NV_\delta^c} \eta_{v,N} \geqslant m_N + z) = 0\,.$$

Applying (34) of Lemma 3.8 to the set $V_N \setminus (NA \cup NV_\delta^c)$ and sending $\delta \searrow 0$ completes the proof of the corollary. $\square$

Before proceeding to the proof of Proposition 4.1, we show that Proposition 2.2 follows directly from it and Corollary 4.2. Recall that $g(\cdot)$ is a function with $g(K) \to \infty$ as $K \to \infty$.

Proof of Proposition 2.2 (assuming Proposition 4.1 and Corollary 4.2). Display (10) is simply a reformation of the first equality in Proposition 4.1. In order to prove (11), it suffices to consider the case when A is an open box. Using the second inequality in Proposition 4.1 and Corollary 4.2, we obtain

$$\lim_{K\to\infty} \lim_{N\to\infty} \frac{\mathrm{e}^{\sqrt{2\pi}(x_K + g(K))}}{g(K) + x_K} \mathbb{P}(\max_{v\in V_{N/K}} \eta_{v,N/K} \geqslant m_{N/K} + g(K) + x_K, Kv_*/N \in A) = \alpha^* \int_A \psi(y)dy\,.$$

Combining this with (10), the desired equality (11) follows by Bayes' formula. $\square$

In order to prove Proposition 4.1, we will study a sparse version of the lattice V_N. Consider $0 < \delta < 1/100$ chosen independently of the other constants. Let $V_N' \subseteq V_N$ be a box in the center of V_N with side length $N' = (1-2\delta)N$. Let L, $\tilde{L}$ and h be integer-valued functions of z, with $h = L/\tilde{L}$, that satisfy

$$\tilde{L} \geqslant 2^{z^4}, \quad h \leqslant \log z, \text{ and } h \to_{z\to\infty} \infty\,. \tag{40}$$

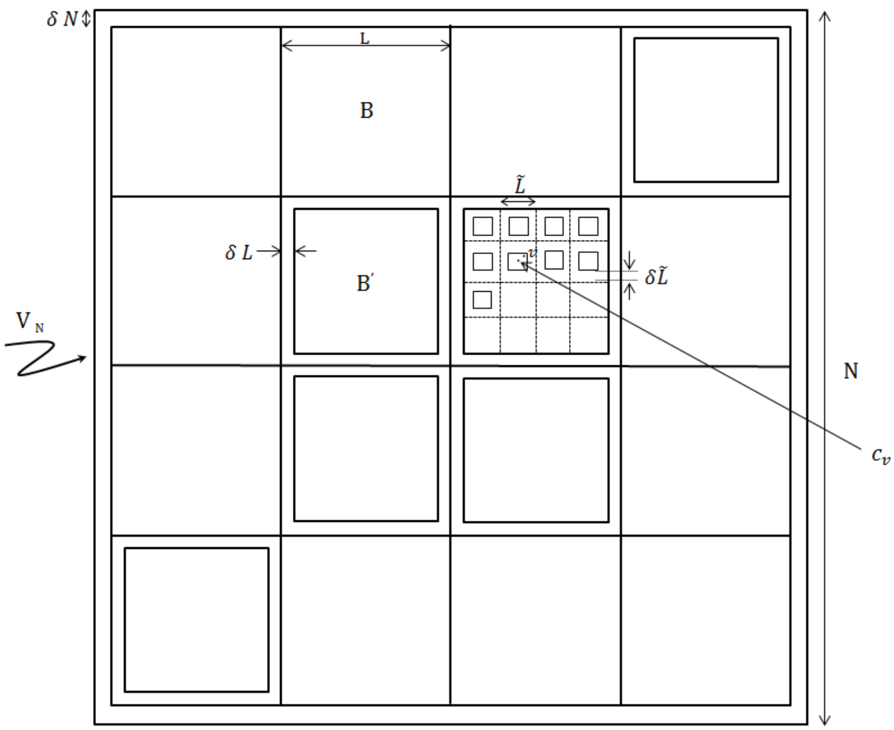

Figure 2: The boxes $B \in \mathcal{B}_N$, B', $\tilde{B}_{v,N}$

Besides (40), the only assumption we impose on L, $\tilde{L}$ and h is that they do not depend on N. In particular, in this section, when taking multiple limits, we will let $N \to \infty$ before taking other limits. (Note that the number of boxes with side length L will go to infinity before L increases. This order differs from that in, e.g., Proposition 2.2, where the lengths of boxes of side length N/K go to infinity before the number of such boxes is allowed to increase.) Throughout the rest of this section, we write

$$n = \log_2 N,\ \ell = \log_2 L,\ \text{ and } \tilde{\ell} = \log_2 \tilde{L}\,.$$

By (40), $\tilde{\ell} \geqslant z^4$ and $\ell \leqslant \tilde{\ell} + \log_2 \log z$.

Let $\mathcal{B}_N$ be the collection of boxes of side length L obtained by partitioning V_N' into $((1-2\delta)N/L)^2$ sub-boxes. For every $B \in \mathcal{B}_N$, let $B' \subseteq B$ be the box in the center of B with side length $(1-2\delta)L$, and let $\tilde{\mathcal{B}}_B$ be the collection of $((1-2\delta)L/\tilde{L})^2$ boxes of side length $\tilde{(}1-2\delta)L$ placed inside B' such that every two boxes are at least $2\delta\tilde{L}$ distance apart. (This collection is obtained by removing from B a grid-patterned set of width $2\delta\tilde{L}$.) Set $\tilde{\mathcal{B}}_N = \cup_{B \in \mathcal{B}_N} \tilde{\mathcal{B}}_B$ and, for each $\tilde{B} \in \tilde{\mathcal{B}}_N$, denote by $c_{\tilde{B}}$ the center of $\tilde{B}$. Furthermore, for each $v \in V_N$, we denote by $B_{v,N}$ and $\tilde{B}_{v,N}$ the boxes in $\mathcal{B}_N$ and $\tilde{\mathcal{B}}_N$ that contain v (if they exist), respectively. Write $\tilde{V}_N = \cup_{\tilde{B} \in \tilde{\mathcal{B}}_N} \tilde{B}$. Finally, for $v \in \tilde{V}_N$, denote by $c_v = c_{\tilde{B}_{v,N}}$ the center of the $\tilde{B}$-box that contains v. Proposition 4.1 follows immediately from the following result and Lemma 3.8, by sending $\delta \searrow 0$.

Proposition 4.3. *For any $0 < \delta \leqslant 1/100$, there exists a constant $\alpha_\delta^* > 0$ such that*

$$\lim_{z \to \infty} \limsup_{N \to \infty} |z^{-1} \mathrm{e}^{\sqrt{2\pi} z} \mathbb{P}(\max_{v \in \tilde{V}_N} \eta_{v,N} \geqslant m_N + z) - \alpha_\delta^*| = 0\,.$$

Furthermore, there exists a continuous function $\psi_\delta : [\delta, 1-\delta]^2 \mapsto (0,\infty)$, with $\int_{[\delta,1-\delta]^2} \psi_\delta(x)dx = 1$, and a continuous function $\psi : (0,1)^2 \mapsto (0,\infty)$, with $\psi_\delta(x) \to \psi(x)$ uniformly in x on closed sets as $\delta \searrow 0$, such that, for any open set $A \subseteq [\delta, 1-\delta]^2$,

$$\lim_{z\to\infty} \limsup_{N\to\infty} |z^{-1}\mathrm{e}^{\sqrt{2\pi}z}\mathbb{P}(\max_{v\in NA\cap\tilde{V}_N} \eta_{v,N} \geqslant m_N + z) - \alpha^*_\delta \int_A \psi_\delta(x)dx| = 0\,.$$

The rest of the section is devoted to the proof of Proposition 4.3. In what follows, we consider $\delta > 0$ to be fixed and suppress any dependence on δ in the notation except in cases where it is important to stress the dependence.

4.1 A mixture of MBRW and GFF

In the proof of Proposition 4.3, we will approximate the GFF by a mixture of a MBRW (in coarse scales) and a copy of the GFF (in fine scales). The approximation consists of two main steps. First, in analogy with the coarse-fine decomposition introduced in Section 2, but employing different scales, we write the GFF as a sum of two independent Gaussian fields. The "fine" field will consist of independent copies of the GFF in smaller boxes, while the "coarse" field will be approximated by a piecewise constant Gaussian field. In the second step, we then further approximate the coarse field by a MBRW.

Step 1. For $v \in \tilde{V}_N$, define (in analogy with (2), except that box sizes are different)

$$\begin{aligned} &X_{v,N} = \mathbb{E}(\eta_{v,N} \mid \mathcal{F}_{\partial B_{v,N}}) \text{ and } Y_{v,N} = \eta_{v,N} - X_{v,N}\,, \\ &\tilde{\eta}_{v,N} = X_{c_v,N} + Y_{v,N}\,. \end{aligned} \tag{41}$$

Note that the process $\mathcal{Y}_B = \{Y_{v,N} : v \in B\}$ is distributed as a GFF on B with Dirichlet boundary data. Moreover, as in the decomposition into the coarse and fine fields in Section 2,

$$\{X_{v,N} : v \in \tilde{V}_n\} \text{ is independent of } \{Y_{v,N} : v \in \tilde{V}_N\}\,, \text{ and } \{\mathcal{Y}_B : B \in \mathcal{B}_N\} \text{ are independent}\,. \tag{42}$$

We first show that the limiting right tail for the maximum of $\{\eta_{\cdot,N}\}$ can be approximated by that of $\{\tilde{\eta}_{\cdot,N}\}$. We start with the following preparatory lemma. This is the only place where the assumption $h \to \infty$ is used.

Lemma 4.4. *There exists $N_0 = N_0(z, \ell, \tilde{\ell})$ and ε_z with $\varepsilon_z \searrow_{z\to\infty} 0$ such that, for all $u, v \in \tilde{V}_N$ and all $N > N_0$,*

$$(1 - \varepsilon_z/\log N)^2 \mathbb{E}X_{u,N}X_{v,N} \leqslant \mathbb{E}X_{c_u,N}X_{c_v,N} \leqslant (1 + \varepsilon_z/\log N)^2 \mathbb{E}X_{u,N}X_{v,N}\,.$$

Proof. We first consider the case when u and v belong to different boxes in $\mathcal{B}_N$. Setting $d_{u,v} = \|u - v\|_2$, we have $d_{u,v} \gtrsim L$. In addition, by the independence in (42),

$$\mathbb{E}X_{u,N}X_{v,N} = \mathbb{E}\eta_{u,N}\eta_{v,N} \text{ and } \mathbb{E}X_{c_u,N}X_{c_v,N} = \mathbb{E}\eta_{c_u,N}\eta_{c_v,N}\,.$$

Let $H_{v,N}$ and $H_{c_v,N}$ be the exit measures on ∂V_N for random walks started at v and c_v, respectively. By [12, Proposition 8.1.4], $\|H_{v,N} - H_{c_v,N}\|_{\mathrm{TV}} \lesssim \tilde{L}/N$, where $\|\mu - \nu\|_{\mathrm{TV}}$ denotes the total variation distance between measures μ and ν. Combined with [12, Lemma 4.6.2], this implies

$$|\mathbb{E}\eta_{u,N}\eta_{v,N} - \mathbb{E}\eta_{c_u,N}\eta_{c_v,N}| \lesssim \frac{\tilde{L}}{N} + \log(1 + \frac{\tilde{L}}{d_{u,v}}) \lesssim \frac{\tilde{L}}{d_{u,v}} \lesssim \frac{\tilde{L}}{L}\frac{\log(\frac{N}{d_{u,v}}) \vee 2)}{\log N} \asymp \frac{\tilde{L}}{L}\frac{\mathbb{E}X_{u,N}X_{v,N}}{\log N}\,. \tag{43}$$

Since $EX_{u,N}X_{v,N} = E\eta_{u,N}\eta_{v,N}$, the last display demonstrates the lemma, when u, v belong to different boxes in $\mathcal{B}_N$, by choosing $\varepsilon_z = C\tilde{L}/L$, for some fixed, absolute constant C.

We next consider $u, v \in B$ for a given $B \in \mathcal{B}_N$. Let H'_v and H'_{c_v} be the exit measures on ∂B for random walks started at v and c_v, respectively. By [12, Proposition 8.1.4], $\|H'_v - H'_{c_v}\|_{\mathrm{TV}} \lesssim \tilde{L}/L$. Combined with [12, Lemma 4.6.2], this implies that

$$|\mathbb{E}X_{u,N}X_{v,N} - \mathbb{E}X_{c_u,N}X_{c_v,N}| \leqslant |\mathbb{E}\eta_{u,N}\eta_{v,N} - \mathbb{E}\eta_{c_u,N}\eta_{c_v,N}| + |\mathbb{E}Y_{u,N}Y_{v,N} - \mathbb{E}Y_{c_u,N}Y_{c_v,N}|$$
$$\lesssim \tilde{L}/N + \tilde{L}/L \lesssim \tilde{L}/L\,.$$

Furthermore,

$$\mathbb{E}X_{u,N}X_{v,N} = \mathbb{E}\eta_{u,N}\eta_{v,N} - \mathbb{E}Y_{u,N}Y_{v,N} = \log N - O(\log L)\,.$$

Together, the last two displays imply that

$$|\mathbb{E}X_{u,N}X_{v,N} - \mathbb{E}X_{c_u,N}X_{c_v,N}| \lesssim (\tilde{L}/L)\cdot(\mathbb{E}X_{u,N}X_{v,N}/\log N)\,.$$

Setting $\varepsilon_z = C\tilde{L}/L$, with C a fixed absolute constant, demonstrates the lemma when u, v belong to the same box in $\mathcal{B}_N$. $\square$

We next compare the maxima of $\eta_{\cdot,N}$ and $\tilde{\eta}_{\cdot,N}$.

Lemma 4.5. *There exist δ_z, with $\delta_z \searrow_{z\to\infty} 0$, such that*

$$\lim_{z\to\infty}\liminf_{N\to\infty}\frac{\mathbb{P}(\max_{v\in\tilde{V}_N}\eta_{v,N} \geqslant m_N + z)}{\mathbb{P}(\max_{v\in\tilde{V}_N}\tilde{\eta}_{v,N} \geqslant m_N + z + \delta_z)} \geqslant 1, \tag{44}$$

$$\lim_{z\to\infty}\limsup_{N\to\infty}\frac{\mathbb{P}(\max_{v\in\tilde{V}_N}\eta_{v,N} \geqslant m_N + z)}{\mathbb{P}(\max_{v\in\tilde{V}_N}\tilde{\eta}_{v,N} \geqslant m_N + z - \delta_z)} \leqslant 1\,. \tag{45}$$

Proof. Choose ε_z as in Lemma 4.4. For $v \in \tilde{V}_N$, define

$$\zeta_{v,N} = (1 - \varepsilon_z/\log N)X_{v,N} + Y_{v,N} + \sqrt{\varepsilon_z}\bar{\phi}_{v,N},$$

where $\bar{\phi}_{v,N}$ are independent Gaussian variables with variances such that $\operatorname{Var}\zeta_{v,N} = \operatorname{Var}\tilde{\eta}_{v,N}$. Because of Lemma 4.4, $\operatorname{Var}\bar{\phi}_{v,N}$ is bounded uniformly in v and N. Using Lemma 4.4 again implies

$$\mathbb{E}\zeta_{u,N}\zeta_{v,N} \leqslant \mathbb{E}\tilde{\eta}_{u,N}\tilde{\eta}_{v,N} \text{ for all } u, v \in \tilde{V}_N\,. \tag{46}$$

Combined with Lemma 3.3, this implies

$$\mathbb{P}(\max_{u\in\tilde{V}_N}\zeta_{u,N} \geqslant \lambda) \geqslant \mathbb{P}(\max_{u\in\tilde{V}_N}\tilde{\eta}_{u,N} \geqslant \lambda) \text{ for all } \lambda \in \mathbb{R}\,. \tag{47}$$

Since $\zeta_{u,N} = \eta_{u,N} - \varepsilon_z X_{u,N}/\log N + \sqrt{\varepsilon_z}\bar{\phi}_{u,N}$, it follows from this that, for all z large enough so that $10\varepsilon_z < \varepsilon_z^{1/4}$,

$$\mathbb{P}(\max_{u\in\tilde{V}_N}\zeta_{u,N} \geqslant m_N + z) \leqslant \mathbb{P}(\max_{u\in\tilde{V}_N}(\eta_{u,N} + \sqrt{\varepsilon_z}\bar{\phi}_{u,N}) \geqslant m_N + z - \varepsilon_z^{1/4}) + \mathbb{P}(\max_{u\in\tilde{V}_N}X_{u,N} \geqslant 10\log N)\,.$$

An identical proof to that of Lemma 3.9 (applied to the maximum in $\tilde{V}_N$, as opposed to V_N) shows that, for $\varepsilon > 0$ and $x > 0$,

$$\mathbb{P}(\max_{u\in\tilde{V}_N} \eta_{u,N} + \varepsilon\bar{\phi}_{u,N} \geqslant m_N + x) \leqslant \mathbb{P}(\max_{u\in\tilde{V}_N} \eta_{u,N} \geqslant m_N + x - \sqrt{\varepsilon})(1 + O(\mathrm{e}^{-C^{-1}\varepsilon^{-1}}))\,.$$

Substituting $\varepsilon = \varepsilon_z^{1/2}$ and $x = z - \varepsilon_z^{1/4} > 0$, for large z, and using the union bound

$$\mathbb{P}(\max_{u\in\tilde{V}_N} X_{u,N} \geqslant 10\log N) \leqslant N^2 \max_{u\in V_N} \mathbb{P}(X_{u,N} \geqslant 10\log N) = O(N^{-4}) \tag{48}$$

implies that

$$\mathbb{P}(\max_{u\in\tilde{V}_N} \zeta_{u,N} \geqslant m_N + z) \leqslant O(N^{-4}) + \mathbb{P}(\max_{u\in\tilde{V}_N} \eta_{u,N} \geqslant m_N + z - 2\varepsilon_z^{1/4})(1 + O(\mathrm{e}^{-C^{-1}\varepsilon_z^{-1/2}}))\,. \tag{49}$$

Together, (47) and (48) imply (44), with $\delta_z = 2\varepsilon_z^{1/4}$. (We have used the result that, for fixed z, the numerator in (44) is bounded below by a positive function of z, as $N \to \infty$; this can be shown by, e.g., an easy adaptation of the argument in [7, Theorem 1.1].)

We next turn to the proof of (45), which is similar in spirit. For every $v \in \tilde{V}_N$, define

$$\hat{\eta}_{v,N} = (1 - \varepsilon_z/\log N)X_{c_v,N} + Y_{v,N} + \sqrt{\varepsilon_z}\hat{\phi}_{v,N}\,,$$

where $\hat{\phi}_{v,N}$ are independent Gaussian variables with variances chosen so that $\operatorname{Var}\hat{\eta}_{v,N} = \operatorname{Var}\eta_{v,N}$. We see that $\operatorname{Var}\hat{\phi}_{v,N}$ is bounded uniformly in N and v, by Lemma 4.4. By this lemma,

$$\mathbb{E}\hat{\eta}_{u,N}\hat{\eta}_{v,N} \leqslant \mathbb{E}\eta_{u,N}\eta_{v,N} \text{ for all } u, v \in \tilde{V}_N\,.$$

An application of Lemma 3.3 implies that

$$\mathbb{P}(\max_{u\in\tilde{V}_N} \hat{\eta}_{u,N} \geqslant \lambda) \geqslant \mathbb{P}(\max_{u\in\tilde{V}_N} \eta_{u,N} \geqslant \lambda) \text{ for all } \lambda \in \mathbb{R}\,. \tag{50}$$

Clearly, with probability $1 - O(N^{-1})$,

$$\max \hat{\eta}_{v,N} \leqslant \max(X_{c_{\tilde{B}},N} + Y_{v,N} + \sqrt{\varepsilon_z}\hat{\phi}_{v,N}) = \max_v(\tilde{\eta}_{v,N} + \sqrt{\varepsilon_z}\hat{\phi}_{v,N})\,. \tag{51}$$

By (46), for any $\Gamma \subseteq \tilde{V}_N$,

$$\mathbb{P}(\max_{v\in\Gamma} \tilde{\eta}_{v,N} \geqslant \lambda) \leqslant \mathbb{P}(\max_{v\in\Gamma} \bar{\eta}_{v,N} \geqslant \lambda) = \mathbb{P}(\max_{v\in\Gamma}(1 - \varepsilon_z/\log N)\eta_{v,N} + \sqrt{\varepsilon_z}\hat{\phi}_{v,N} \geqslant \lambda)\,.$$

Repeating the argument in the proof of Lemma 3.9, we obtain

$$\mathbb{P}(\max_{u\in\tilde{V}_N} \tilde{\eta}_{u,N} + \sqrt{\varepsilon_z}\hat{\phi}_{N,z} \geqslant m_N + z) \leqslant \mathbb{P}(\max_{u\in\tilde{V}_N} \tilde{\eta}_{u,N} \geqslant m_N + z - 2\varepsilon_z^{1/4}) + z\mathrm{e}^{-\sqrt{2\pi}z}O(\mathrm{e}^{C^{-1}\varepsilon_z^{-1/2}})\,.$$

Together with (50) and (51), this completes the proof of the lemma. □

Step 2. Define

$$\Xi_N = \{c_{\tilde{B}} : \tilde{B} \in \tilde{\mathcal{B}}_N\}. \tag{52}$$

We next approximate $\{X_{v,N} : v \in \Xi_N\}$ by a MBRW, by using the notation of Section 3.1. For $v \in \Xi_N$, define

$$S_{v,N} = \textstyle\sum_{j=\ell}^{n} \sum_{B \in \mathfrak{B}_j(v)} \phi_{N,j,B} \,.$$

Note that, for $B \in \mathcal{B}$, the process $\{\xi_{v,N} - S_{v,N} : v \in B \cap \Xi_N\}$ is a MBRW (projected onto Ξ_N) that is defined with respect to the box B, except that the torus wraps around with respect to V_N, rather than B. However, since $\Xi_N \cap B$ is distance δL away from ∂B, it is clear that this modification only changes the covariance for any pair of vertices by up to an additive constant $C = C_\delta$, which depends only on δ. Therefore, by Lemma 3.1,

$$|\mathrm{Cov}(\xi_{u,N} - S_{u,N}, \xi_{v,N} - S_{v,N}) - \mathrm{Cov}(Y_{u,N}, Y_{v,N})| \leqslant C_\delta \text{ for all } u, v \in \Xi_N \,.$$

Lemma 3.1 also implies

$$|\mathrm{Cov}(\xi_{u,N}, \xi_{v,N}) - \mathrm{Cov}(\eta_{u,N}, \eta_{v,N})| \leqslant C_\delta \text{ for all } u, v \in \Xi_N \,.$$

Together, these two inequalities imply

$$|\mathrm{Cov}(S_{u,N}, S_{v,N}) - \mathrm{Cov}(X_{u,N}, X_{v,N})| \leqslant C_\delta \text{ for all } u, v \in \Xi_N \,. \tag{53}$$

Next, let r be an integer to be specified and, for $v \in \Xi_N$, define

$$S^{\mathrm{up}}_{v,N,r} = \textstyle\sum_{j=\ell}^{n-r} \sum_{B \in \mathfrak{B}_j(v)} \phi_{N,j,B} \text{ and } S^{\mathrm{lw}}_{v,N,r} = \sum_{j=\ell+r}^{n} \sum_{B \in \mathfrak{B}_j(v)} \phi_{N,j,B} \,. \tag{54}$$

Also, define

$$X^{\mathrm{up}}_{v,N,r} = S^{\mathrm{up}}_{v,N,r} + \phi_{v,N,r} \text{ and } X^{\mathrm{lw}}_{v,N,r} = S^{\mathrm{lw}}_{v,N,r} + a_{v,N,r}\phi \,, \tag{55}$$

where $\phi_{v,N,r}$ are independent mean zero Gaussian variables so that $\mathrm{Var}\, X^{\mathrm{up}}_{v,N,r} = \mathrm{Var}\, X_{v,N}$, and ϕ is a standard independent Gaussian variable, with $a_{v,N,r}$ chosen so that $\mathrm{Var}\, X^{\mathrm{lw}}_{v,N,r} = \mathrm{Var}\, X_{v,N}$.

Lemma 4.6. *There exists $r = r_\delta$, depending only on δ, such that, for all $N \in \mathbb{N}$,*

$$\mathbb{E} X^{\mathrm{up}}_{v,N,r} X^{\mathrm{up}}_{u,N,r} \leqslant \mathbb{E} X_{u,N} X_{v,N} \leqslant \mathbb{E} X^{\mathrm{lw}}_{v,N,r} X^{\mathrm{lw}}_{u,N,r} \textit{ for all } u, v \in \Xi_N \,.$$

Proof. For $u, v \in \Xi_N$, with $0 < d_N(u, v) \leqslant 2^{-r} N$,

$$\mathbb{E} X^{\mathrm{up}}_{v,N,r} X^{\mathrm{up}}_{u,N,r} = \mathbb{E} S^{\mathrm{up}}_{v,N,r} S^{\mathrm{up}}_{u,N,r} \leqslant \mathbb{E} S_{v,N} S_{u,N} - r/2 \,.$$

Employing (53), we can choose r, depending on δ, such that $\mathbb{E} S_{v,N} S_{u,N} \leqslant \mathbb{E} X_{v,N} X_{u,N} + r/2$ and therefore $\mathbb{E} X^{\mathrm{up}}_{v,N,r} X^{\mathrm{up}}_{u,N,r} \leqslant \mathbb{E} X_{v,N} X_{u,N}$. If $d_N(u, v) > 2^{-r} N$, then

$$\mathbb{E} X^{\mathrm{up}}_{v,N,r} X^{\mathrm{up}}_{u,N,r} = \mathbb{E} S^{\mathrm{up}}_{v,N,r} S^{\mathrm{up}}_{u,N,r} = 0 \leqslant \mathbb{E} X_{u,N} X_{v,N} \,.$$

This demonstrates the first inequality. The second inequality follows in the same manner, by considering the Gaussian distances between two vertices (that is, $\sqrt{\mathbb{E}(X_{u,N} - X_{v,N})^2}$). $\square$

In light of the preceding lemma, we fix the value of r in what follows and drop it from the notation. In particular, we write $a_{v,N} = a_{v,N,r}$ for $a_{v,N,r}$ as in (55). By Lemma 3.1, for some constant C independent of N,

$$a_{v,N}^2 \leqslant C\,. \tag{56}$$

For $v \in \tilde{V}_N$, we write $v = \bar{v} + \hat{v}$, with $\bar{v} = c_{B_v}$ and $\hat{v} = v - \bar{v}$. Applying [12, Proposition 8.1.4] and [12, Lemma 4.6.2], as in the proof of Lemma 4.4, we see that

$$a_{(xN+\hat{v}),N} \to g(x,\hat{v}) \text{ for all } (x,\hat{v}) \in [\delta, 1-\delta]^2 \times [-L/2, L/2]^2\,, \tag{57}$$

where $g : [0,1]^2 \times [-L/2, L/2]^2 \mapsto [0,\infty)$ is a function that is continuous in the first coordinate. Finally, for every $v \in \tilde{V}_N$, define

$$\eta_{v,N}^{\mathrm{up}} = X_{c_v,N}^{\mathrm{up}} + Y_{v,N} \text{ and } \eta_{v,N}^{\mathrm{lw}} = X_{c_v,N}^{\mathrm{lw}} + Y_{v,N}\,.$$

By Lemma 4.6 and Lemma 3.3, for all $N \in \mathbb{N}$ and $\lambda \in \mathbb{R}$,

$$\mathbb{P}(\max_{v\in\tilde{V}_N} \eta_{v,N}^{\mathrm{lw}} \geqslant \lambda) \leqslant \mathbb{P}(\max_{v\in\tilde{V}_N} \tilde{\eta}_{v,N} \geqslant \lambda) \leqslant \mathbb{P}(\max_{v\in\tilde{V}_N} \eta_{v,N}^{\mathrm{up}} \geqslant \lambda)\,. \tag{58}$$

Therefore, by Lemma 4.5, (34) (applied to $V_N \setminus \tilde{V}_N$) and (27),

$$\mathbb{P}(\max_{v\in\tilde{V}_N} \eta_{v,N}^{\mathrm{up}} \geqslant m_N + z) \gtrsim z\mathrm{e}^{-\sqrt{2\pi}z}\,. \tag{59}$$

The fields η^{up} and η^{lw} are the approximations of the GFF that will be employed in the proof of Proposition 4.3.

4.2 Enumeration of the large clusters determines the limiting tail

Write $\gamma = \sqrt{2\log 2/\pi}$ and let Ξ_N be as in (52). For $v \in \Xi_N$, we set $n_{v,N} = \gamma^{-2}\operatorname{Var} X_{v,N}$; clearly, $n_{v,N} = (1+o(1))n$. For convenience, we now view each $X_{v,N}$ as the value at time $n_{v,N}$ of a Brownian motion with variance rate γ^2. More precisely, we associate to each Gaussian variable $\phi_{N,j,B}$ in (54) an independent Brownian motion, with variance rate γ^2, that runs for 2^{-2j} time units and ends at the value $\phi_{N,j,B}$. In the same manner, we associate to $\phi_{v,N,r}$ in (55) a Brownian motion of variance rate γ^2 that runs for $\gamma^{-2}\operatorname{Var}\phi_{v,N,r}$ time units and ends at the value $\phi_{v,N,r}$. For the Gaussian variable ϕ in (55), we employ a standard Brownian motion $\{W_t : t \in [0,1]\}$, with $W_1 = \phi$. When adding $a_{v,N,r}\phi$ to a vertex $v \in \Xi_N$, as in (55), we consider the Brownian motion $W_{v,N,t} = \gamma W_{\gamma^{-2}a_{v,N}^{-2}t}$, with $t \in [0, \gamma^2 a_{v,N}^2]$.

We now define a Brownian motion $\{X_{v,N}^{\mathrm{up}}(t) : 0 \leqslant t \leqslant n_{v,N}\}$ ($\{X_{v,N}^{\mathrm{lw}}(t) : 0 \leqslant t \leqslant n_{v,N}\}$) by concatenating all of the previous Brownian motions associated with v in an order so that the sizes of the involved boxes are non-increasing (where we view $\phi_{N,j,B}$ as associated with a box of size 0 and ϕ as associated with a box of size ∞). From our construction, we see that $X_{v,N}^{\mathrm{up}}(n_{v,N}) = X_{v,N}^{\mathrm{up}}$

and $X^{\mathrm{lw}}_{v,N}(n_{v,N}) = X^{\mathrm{lw}}_{v,N}$. We write $n^* = \max_{v\in\Xi_N} n_{v,N}$ and define

$$\begin{aligned}
E^{\mathrm{up}}_{v,N}(z) &= \{X^{\mathrm{up}}_{v,N}(t) \leqslant z + \frac{m_N}{n}t \text{ for all } 0 \leqslant t \leqslant n_{v,N}, \text{ and } \max_{u\in\tilde{B}_v} \eta^{\mathrm{up}}_{u,N} \geqslant m_N + z\}\,,\\
F^{\mathrm{up}}_{v,N}(z) &= \{X^{\mathrm{up}}_{v,N}(t) \leqslant z + \frac{m_N}{n}t + 10(\log(t\wedge(n^*-t)))_+ + z^{1/20}\\
&\qquad \text{for all } 0 \leqslant t \leqslant n_{v,N}, \text{ and } \max_{u\in\tilde{B}_v} \eta^{\mathrm{up}}_{u,N} \geqslant m_N + z\}\,,\\
G^{\mathrm{up}}_{N}(z) &= \bigcup_{v\in\Xi_N}\bigcup_{0\leqslant t\leqslant n_{v,N}} \{X^{\mathrm{up}}_{v,N}(t) > z + \frac{m_N}{n}t + 10(\log(t\wedge(n^*-t)))_+ + z^{1/20}\}\,,\\
E^{\mathrm{lw}}_{v,N}(z) &= \{X^{\mathrm{lw}}_{v,N}(t) \leqslant z + \frac{m_N}{n}t \text{ for all } 0 \leqslant t \leqslant n_{v,N}, \text{ and } \max_{u\in\tilde{B}_v} \eta^{\mathrm{lw}}_{u,N} \geqslant m_N + z\}\,.
\end{aligned} \tag{60}$$

Also define

$$\Lambda^{\mathrm{up}}_{N,z} = \sum_{v\in\Xi_N} \mathbf{1}_{E^{\mathrm{up}}_{v,N}(z)}\,, \Gamma^{\mathrm{up}}_{N,z} = \sum_{v\in\Xi_N} \mathbf{1}_{F^{\mathrm{up}}_{v,N}(z)}\,, \Lambda^{\mathrm{lw}}_{N,z} = \sum_{v\in\Xi_N} \mathbf{1}_{E^{\mathrm{lw}}_{v,N}(z)}\,,$$

and, for a box $A \subseteq [\delta, 1-\delta]^2$, define

$$\Lambda^{\mathrm{lw}}_{N,z}(A) = \sum_{v\in\Xi_N\cap NA} \mathbf{1}_{E^{\mathrm{lw}}_{v,N}(z)}\,.$$

In words, the random variable $\Lambda^{\mathrm{up}}_{N,z}$ counts the number of clusters whose "backbone" path $X^{\mathrm{up}}_{v,N}(\cdot)$ stays below a linear path connecting z to roughly $m_N + z$, so that one of its "neighbors" achieves a terminal value that is at least $m_N + z$; the random variable $\Gamma^{\mathrm{up}}_{N,z}$ similarly counts clusters whose backbone is constrained to stay below a slightly "upward bent" curve.

Note that it follows from their definitions that, for fixed v, the processes $X^{\mathrm{up}}_{v,N}(\cdot)$ and $X^{\mathrm{lw}}_{v,N}(\cdot)$ have the same distribution. Furthermore, for any fixed $v \in \Xi_N$, and in particular for $c_v = v$,

$$\max_{u\in\tilde{B}_v} \eta^{\mathrm{up}}_{u,N} = X^{\mathrm{up}}_{c_v,N} + \max_{u\in\tilde{B}_v} Y_{u,N}\,, \quad \max_{u\in\tilde{B}_v} \eta^{\mathrm{lw}}_{u,N} = X^{\mathrm{lw}}_{c_v,N} + \max_{u\in\tilde{B}_v} Y_{u,N}\,,$$

where $\max_{u\in\tilde{B}_v} Y_{u,N}$ is independent of both $X^{\mathrm{up}}_{c_v,N}$ and $X^{\mathrm{lw}}_{c_v,N}$. Therefore, for any fixed $v \in \Xi_N$, the events $E^{\mathrm{up}}_{v,N}(z)$ and $E^{\mathrm{lw}}_{v,N}(z)$ have the same probability, which implies that

$$\mathbb{E}\Lambda^{\mathrm{lw}}_{N,z} = \mathbb{E}\Lambda^{\mathrm{up}}_{N,z}\,. \tag{61}$$

The main result of this subsection is the following proposition.

Proposition 4.7. *There exist $\delta_z \geqslant 0$ with $\delta_z \searrow_{z\to\infty} 0$ such that, for any open box $A \subseteq [\delta, 1-\delta]^2$,*

$$\lim_{z\to\infty}\limsup_{N\to\infty} \frac{\mathbb{P}(\max_{v\in\tilde{V}_N\cap NA} \eta_{v,N} \geqslant m_N + z)}{\mathbb{E}\Lambda^{\mathrm{lw}}_{N,z-\delta_z}(A)} \leqslant 1 \leqslant \lim_{z\to\infty}\liminf_{N\to\infty} \frac{\mathbb{P}(\max_{v\in\tilde{V}_N\cap NA} \eta_{v,N} \geqslant m_N + z)}{\mathbb{E}\Lambda^{\mathrm{lw}}_{N,z+\delta_z}(A)}\,. \tag{62}$$

In particular,

$$\lim_{z\to\infty}\limsup_{N\to\infty} \frac{\mathbb{P}(\max_{v\in\tilde{V}_N} \eta_{v,N} \geqslant m_N + z)}{\mathbb{E}\Lambda^{\mathrm{lw}}_{N,z-\delta_z}} \leqslant 1 \leqslant \lim_{z\to\infty}\liminf_{N\to\infty} \frac{\mathbb{P}(\max_{v\in\tilde{V}_N} \eta_{v,N} \geqslant m_N + z)}{\mathbb{E}\Lambda^{\mathrm{lw}}_{N,z+\delta_z}}\,. \tag{63}$$

The proof of (62) does not require more work than the proof of (63), but it does involve more notation; for the sake of economy, we therefore only prove (63). The display (63) is an immediate consequence of Lemma 4.5, (58), (61) and the next proposition.

Proposition 4.8. *With notation as above,*

$$\lim_{z\to\infty}\limsup_{N\to\infty}\frac{\mathbb{P}(\max_{v\in\tilde V_N}\eta^{\mathrm{lw}}_{v,N}\geqslant m_N+z)}{\mathbb{E}\Lambda^{\mathrm{lw}}_{N,z}}=\lim_{z\to\infty}\liminf_{N\to\infty}\frac{\mathbb{P}(\max_{v\in\tilde V_N}\eta^{\mathrm{lw}}_{v,N}\geqslant m_N+z)}{\mathbb{E}\Lambda^{\mathrm{lw}}_{N,z}}=1\,,$$
$$\lim_{z\to\infty}\limsup_{N\to\infty}\frac{\mathbb{P}(\max_{v\in\tilde V_N}\eta^{\mathrm{up}}_{v,N}\geqslant m_N+z)}{\mathbb{E}\Lambda^{\mathrm{up}}_{N,z}}=\lim_{z\to\infty}\liminf_{N\to\infty}\frac{\mathbb{P}(\max_{v\in\tilde V_N}\eta^{\mathrm{up}}_{v,N}\geqslant m_N+z)}{\mathbb{E}\Lambda^{\mathrm{up}}_{N,z}}=1\,.$$

In order to prove Proposition 4.8, we separately derive upper and lower bounds. For these bounds, we consider truncations of the MBRW profile with respect to certain upper and lower curves, as in the definitions of $F^{\mathrm{up}}_{v,N}(z)$ and $E^{\mathrm{lw}}_{v,N}(z)$ in (60). In defining these truncations, the following two requirements are crucial:

(1) The two truncations result asymptotically in the same probability; this will be shown in Lemma 4.10 (the underlying reason being the bounds in Lemma 3.6).

(2) After truncation with respect to the lower curve, the resulting second moment is asymptotically the same as the corresponding first moment; this will be shown in Lemma 4.11. (In the lemma, we will count clusters as opposed to vertices; this leads to an improvement of the bound in [8], and allows us to give precise asymptotics for our tail estimates.)

We first compare $\Lambda^{\mathrm{up}}_{N,z}$ and $\Gamma^{\mathrm{up}}_{N,z}$, and start with the following estimate.

Lemma 4.9. *There exists $N_0=N_0(z,\ell,\tilde\ell,\delta)$ such that, for any $\tilde B_1\neq\tilde B_2\in\tilde{\mathcal{B}}_N$, any $\lambda_1,\lambda_2>0$ and any $N>N_0$,*

$$\begin{aligned}&\mathbb{P}(\max_{u\in\tilde B_1}Y_{u,N}\geqslant \ell m_N/n+\lambda_1,\max_{u\in\tilde B_2}Y_{u,N}\geqslant \ell m_N/n+\lambda_2)\\&\lesssim(\log z)^C\ell^{-3}(\lambda_1+\log\ell)(\lambda_2+\log\ell)\mathrm{e}^{-\sqrt{2\pi}(\lambda_1+\lambda_2)}\mathrm{e}^{-C^{-1}(\lambda_1^2+\lambda_2^2)/\ell}\,,\end{aligned}\tag{64}$$

where $C>0$ is an absolute constant. Moreover,

$$\mathbb{P}(\max_{u\in\tilde B_1}Y_{u,N}\geqslant \ell m_N/n+\lambda_1)\lesssim(\log z)^C\ell^{-3/2}(\lambda_1+\log\ell)\mathrm{e}^{-\sqrt{2\pi}\lambda_1}\mathrm{e}^{-C^{-1}\lambda_1^2/\ell}\,.\tag{65}$$

The constant C in (64) is chosen large enough so as to absorb the correlation between the two events in the display.

Proof. We give a proof for (64) and omit the proof of (65) (which is simpler). Let $\hat B_1$ and $\hat B_2$ be boxes of side length $\tilde L(1-\delta)$ that have the same centers as $\tilde B_1$ and $\tilde B_2$ (and thus $\tilde B_i\subset\hat B_i$). For $u\in\tilde B_i$ (i=1,2), we define

$$\Phi_{u,N}=\mathbb{E}(Y_{u,N}\mid\{Y_{w,N}:w\in\partial\hat B_i\})\text{ and }\Psi_{u,N}=Y_{u,N}-\Phi_{u,N}\,.$$

Clearly, $\{\Psi_{u,N}:u\in\tilde B_1\}$ is independent of $\{\Psi_{u,N}:u\in\tilde B_2\}$. Repeating the computations from Lemma 3.10, we have that, for u,v in the same box $\tilde B_i$,

$$\mathbb{E}(\Phi_{u,N}-\Phi_{v,N})^2\leqslant c(\delta)\frac{|u-v|}{\tilde L(1-2\delta)}\,.\tag{66}$$

Furthermore, as in (6) in the proof of Lemma 2.1, $\operatorname{Var}\Phi_{u,N}$ can be represented as the difference of the variances of GFFs in boxes of side length L and $(1-\delta)\tilde{L}$, which leads to

$$\sigma^2 := \max_{u\in\tilde{B}_1\cup\tilde{B}_2} \operatorname{Var}\Phi_{u,N} \lesssim \log\log z\,, \tag{67}$$

where we have used $h = L/\tilde{L} \leqslant \log z$ and Lemma 3.1. By (66) and Lemma 3.5 (applied to boxes of side length $\tilde{L}(1-2\delta)$), $\mathbb{E}W_i \lesssim 1$, where $W_i := \max_{u\in\tilde{B}_i}\Phi_{u,N}$ for $i\in\{1,2\}$. By the last inequality, (67) and Lemma 3.4, there therefore exists a constant $C>0$ such that

$$\mathbb{P}(W_i \geqslant \lambda) \lesssim \mathrm{e}^{-\lambda^2/(2C\log\log z)}\,. \tag{68}$$

We now write

$$\begin{aligned}
&\mathbb{P}(\max_{u\in\tilde{B}_1} Y_{u,N} \geqslant \ell m_N/n+\lambda_1, \max_{u\in\tilde{B}_2} Y_{u,N} \geqslant \ell m_N/n+\lambda_2)\\
&\lesssim \int_0^\infty \max_{i\in\{1,2\}} \mathbb{P}(W_i\geqslant\lambda)\mathbb{P}(\max_{u\in\hat{B}_1}\Psi_{u,N}\geqslant \ell m_N/n+\lambda_1-\lambda)\mathbb{P}(\max_{u\in\hat{B}_2}\Psi_{u,N}\geqslant \ell m_N/n+\lambda_2-\lambda)d\lambda\\
&=\int_0^\infty \max_{i\in\{1,2\}} \mathbb{P}(W_i\geqslant\lambda)\mathbb{P}(\max_{u\in\hat{B}_1}\Psi_{u,N}\geqslant m_\ell+\bar{c}\log\ell+\lambda_1-\lambda+O(\ell\log n/n))\\
&\qquad\qquad\cdot\mathbb{P}(\max_{u\in\hat{B}_2}\Psi_{u,N}\geqslant m_\ell+\bar{c}\log\ell+\lambda_2-\lambda+O(\ell\log n/n))d\lambda\,,
\end{aligned}$$

where $\bar{c} = 3\sqrt{2/\pi}/4$; we have used (1) in the equality. From the last estimate, (33) in Lemma 3.8 (applied in the boxes $\hat{B}_1, \hat{B}_2$ instead of V_N) and (68), it follows that

$$\begin{aligned}
&\mathbb{P}(\max_{u\in\tilde{B}_1} Y_{u,N} \geqslant \ell m_N/n+\lambda_1, \max_{u\in\tilde{B}_2} Y_{u,N} \geqslant \ell m_N/n+\lambda_2)\\
&\lesssim \int_0^\infty \mathrm{e}^{-\lambda^2/(2C\log\log z)}\ell^{-3}(\lambda_1+\log\ell)(\lambda_2+\log\ell)\mathrm{e}^{-\sqrt{2\pi}(\lambda_1+\lambda_2-2\lambda)}\mathrm{e}^{-C^{-1}((\lambda_1-\lambda)^2+(\lambda_2-\lambda)^2)/\ell}d\lambda\\
&\lesssim \ell^{-3}(\lambda_1+\log\ell)(\lambda_2+\log\ell)\mathrm{e}^{-C^{-1}(\lambda_1^2+\lambda_2^2)/\ell}\int_0^\infty \mathrm{e}^{-\lambda^2/(2C\log\log z)}\mathrm{e}^{2(\sqrt{2\pi}+C(\lambda_1+\lambda_2)/\ell)\lambda}d\lambda\\
&\lesssim \ell^{-3}(\lambda_1+\log\ell)(\lambda_2+\log\ell)\mathrm{e}^{-\sqrt{2\pi}(\lambda_1+\lambda_2)}\mathrm{e}^{-\frac{\lambda_1^2+\lambda_2^2}{C\ell}}\mathrm{e}^{2C\log\log z(\sqrt{2\pi}+\frac{C}{\ell}(\lambda_1+\lambda_2))^2}\\
&\quad\times\int_0^\infty \mathrm{e}^{-\frac{(\lambda-2C\log\log z(\sqrt{2\pi}+C(\lambda_1+\lambda_2)/\ell))^2}{2C\log\log z}}d\lambda\\
&\lesssim (\log z)^{C'}\ell^{-3}(\lambda_1+\log\ell)(\lambda_2+\log\ell)\mathrm{e}^{-\sqrt{2\pi}(\lambda_1+\lambda_2)}\mathrm{e}^{-C'^{-1}(\lambda_1^2+\lambda_2^2)/\ell}\,,
\end{aligned}$$

where C' is a large absolute constant. (In the last inequality, we used the assumption that $z^4 \leqslant \ell$.) $\square$

Lemma 4.10. *For $\Lambda^{\mathrm{up}}_{N,z}$ and $\Gamma^{\mathrm{up}}_{N,z}$ as above,*

$$\lim_{z\to\infty}\liminf_{N\to\infty}\frac{\mathbb{E}\Lambda^{\mathrm{up}}_{N,z}}{\mathbb{E}\Gamma^{\mathrm{up}}_{N,z}} = 1\,. \tag{69}$$

(Of course, the lim inf in (69) also implies the same statement, but with lim sup replacing lim inf, since the ratio is always bounded above by 1.)

Proof. To simplify notation, we drop the superscript "up" from the notation in this proof. For any $v \in \Xi_N$, we write $\bar{X}_{v,N}(t) = X_{v,N}(t) - \frac{m_N t}{n}$, and define the probability measure $\mathbb{Q}$ by

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = \mathrm{e}^{-\frac{m_N}{n\gamma^2}\bar{X}_{v,N}(n_{v,N}) - \frac{m_N^2}{2\gamma^2 n^2} n_{v,N}} . \tag{70}$$

Under $\mathbb{Q}$, $\bar{X}_{v,N}(t)$ is a Brownian motion with variance rate γ^2.

We continue to use the notation $\mu_{t,y}(\cdot)$ and $\mu^*_{t,y}(\cdot)$ from (19) (with variance rate $\sigma^2 = \gamma^2$). With a slight abuse of notation, we write $d\mathbb{P}/d\mathbb{Q} = (d\mathbb{P}/d\mathbb{Q})(\bar{X}_{v,N})$. We have

$$\begin{aligned}
\mathbb{P}(F_{v,N}(z) \setminus E_{v,N}(z)) = &\int_z^{z+z^{1/20}} \frac{d\mathbb{P}}{d\mathbb{Q}}(x)\mu^*_{n_{v,N},z}(x)\mathbb{P}(\max_{u \in \tilde{B}_{v,N}} Y_{u,N} \geqslant \ell m_N/n + z - x)dx \\
&+ \int_{-\infty}^{z} \frac{d\mathbb{P}}{d\mathbb{Q}}(x)(\mu^*_{n_{v,N},z}(x) - \mu_{n_{v,N},z}(x))\mathbb{P}(\max_{u \in \tilde{B}_{v,N}} Y_{u,N} \geqslant \ell m_N/n + z - x)dx \\
&\lesssim 4^{-n^*} z^3 (z + \log \ell)(\log z)^C \ell^{-3/2} \mathrm{e}^{-\sqrt{2\pi} z} + \delta_z \mathbb{P}(E_{v,N}(z)) ,
\end{aligned}$$

where $\delta_z \searrow_{z\to\infty} 0$; the last inequality follows for large ℓ by rewriting $\ell m_N/n$ in terms of m_ℓ, and applying Lemmas 3.6 and 4.9. Therefore,

$$\mathbb{E}\Gamma_{N,z} - \mathbb{E}\Lambda_{N,z} \lesssim z^3 (z + \log \ell)(\log z)^C \ell^{-3/2} \mathrm{e}^{-\sqrt{2\pi} z} + \delta_z \mathbb{E}\Lambda_{N,z} . \tag{71}$$

By (59) and Lemma 3.7 (applied with $\beta = z + z^{1/20}$),

$$\mathbb{E}\Gamma_{N,z} \gtrsim z \mathrm{e}^{-\sqrt{2\pi} z} . \tag{72}$$

Together, (71) and (72) imply (69), which completes the proof of the lemma. □

We next estimate the second moment of $\Lambda^{\mathrm{lw}}_{N,z}$.

Lemma 4.11. *With notation as above,*

$$\lim_{z\to\infty} \limsup_{N\to\infty} \frac{\mathbb{E}(\Lambda^{\mathrm{lw}}_{N,z})^2}{\mathbb{E}\Lambda^{\mathrm{lw}}_{N,z}} = 1 . \tag{73}$$

Proof. Recall from (61) that $\mathbb{E}\Lambda^{\mathrm{lw}}_{N,z} = \mathbb{E}\Lambda^{\mathrm{up}}_{N,z}$. Combined with Lemmas 4.10 and (72), this implies

$$\mathbb{E}\Lambda^{\mathrm{lw}}_{N,z} \gtrsim z \mathrm{e}^{-\sqrt{2\pi} z} . \tag{74}$$

So, the main work is to estimate the above second moment, which we rewrite as

$$E(\Lambda^{\mathrm{lw}}_{N,z})^2 = E(\Lambda^{\mathrm{lw}}_{N,z}) + \sum_{v,w \in \Xi_N, v \neq w} \mathbb{P}(E^{\mathrm{lw}}_{v,N}(z) \cap E^{\mathrm{lw}}_{w,N}(z)) . \tag{75}$$

To simplify notation, we drop the superscript lw in the remainder of the proof of the lemma. We also employ the following terminology. Recalling the formula (70), for any $v \in \Xi_N$, we write $\bar{X}_{v,N}(t) = X_{v,N}(t) - m_N t/n$, with $\bar{X}_{v,N}(t) = \bar{X}_{v,N}(n_{v,N})$ for $n_{v,N} \leqslant t \leqslant n$ and $n^* = \max_{v \in \Xi_N} n_{v,N}$. By (56), $|n^* - n_{v,N}| = O(1)$ uniformly in $v \in \Xi_N$. For $v, w \in \Xi_N$, we say that v and w *split* at time

$t_s = n^* - s$, denoted by $v \sim_s w$, if s is the maximal integer such that $\{X_{v,N}(t) - X_{v,N}(t_s) : t_s \leqslant t \leqslant n^*\}$ is independent of $\{X_{w,N}(t) - X_{w,N}(t_s) : t_s \leqslant t \leqslant n^*\}$. (So, for $v \sim_s w$, $|v - w| \asymp 2^s$.)

We will show that, for large z, the sum in (75) is small in comparison with the first term on the right hand side, by decomposing it into three parts, with v and w satisfying $v \sim_s w$, with s restricted to $[n^* - z^{1/10}, n^*]$, $[z^{1/10}, n^* - z^{1/10})$ and $[1, z^{1/10})$, respectively. For $v \sim_s w$, with given s, we will employ the upper bound

$$\begin{aligned}
&\mathbb{P}(E_{v,N}(z) \cap E_{w,N}(z)) \\
=&\mathbb{P}(\bar{X}_{v,N}(t), \bar{X}_{w,N}(t) \leqslant z \text{ for all } t \in [0, n^*]; \max_{u \in \tilde{B}_v} \eta_{u,N}, \max_{u \in \tilde{B}_w} \eta_{u,N} \geqslant m_N + z) \\
=&\sum_{x \leqslant z} \mathbb{P}(\bar{X}_{v,N}(t), \bar{X}_{w,N}(t) \leqslant z \text{ for all } t \in [0, n^*]; \max_{u \in \tilde{B}_v} \eta_{u,N}, \max_{u \in \tilde{B}_w} \eta_{u,N} \geqslant m_N + z; \bar{X}_{v,N}(t_s) \in (x-1, x]) \\
\leqslant&\sum_{x \leqslant z} \mathbb{P}(\bar{X}_{v,N}(t) \leqslant z \text{ for all } t \in [0, t_s], \bar{X}_{v,N}(t_s) \in [x-1, x])\Gamma_{v,x,z,s}\Gamma_{w,x,z,s}\,,
\end{aligned} \tag{76}$$

where, in the above sum, $x \leqslant z$ is a shorthand notation for $x = z, z-1, \ldots$, and

$$\Gamma_{v,x,z,s} := \sup_{\bar{X}_{v,N}(t_s) \in [x-1,x]} \mathbb{P}(\bar{X}_{v,N}(t) \leqslant z \text{ for all } t_s < t \leqslant n^*, \max_{u \in \tilde{B}_v} \eta_{u,N} \geqslant m_N + z \mid \bar{X}_{v,N}(t_s)).$$

In order to estimate the second moment, we first consider the case $v \sim_s w$, with $n^* - z^{1/10} \leqslant s \leqslant n^*$. Here, $|v - w|_2 \asymp_{z,\tilde{L}} N$; therefore, $\tilde{B}_v$ and $\tilde{B}_w$ belong to different boxes in $\mathcal{B}_N$ when N is large and, in particular, $\{Y_{u,N} : u \in \tilde{B}_v\}$ is independent of $\{Y_{u,N} : u \in \tilde{B}_w\}$. By a change of measure that transforms $\bar{X}_{v,N}(\cdot)$ into Brownian motion and by the ballot theorem (see [1, Theorem 1]),

$$\begin{aligned}
\Gamma_{v,x,z,s} &\leqslant \sum_{y \leqslant z} \mathbb{P}(\bar{X}_{v,N}(r) \leqslant z - x \text{ for all } r \in [0, s], \bar{X}_{v,N}(s) \in [y-1-x, y-x])\gamma_{v,y} \\
&\lesssim \sum_{y \leqslant z} \mathrm{e}^{-\frac{\alpha_n^2}{2}s} \mathrm{e}^{-\alpha_n(y-x)/\gamma} \frac{(z-x)(z-y)}{s^{3/2}} \gamma_{v,y}\,,
\end{aligned} \tag{77}$$

where $\alpha_n = m_N/\gamma n$ (with $\gamma = \sqrt{2\log 2/\pi}$, as before) and $\gamma_{v,y} = \mathbb{P}(\max_{u \in \tilde{B}_v} Y_{u,N} \geqslant \ell m_N/N + z - y)$. On the other hand, obviously

$$\begin{aligned}
\mathbb{P}(\bar{X}_{v,N}(t) \leqslant z, \text{ for all } t \in [0, t_s], \bar{X}_{v,N}(t_s) \in [x-1, x]) &\leqslant \mathbb{P}(\bar{X}_{v,N}(t_s) \in [x-1, x]) \\
&\lesssim \frac{1}{\sqrt{n^* - s}} \mathrm{e}^{-\frac{(\alpha_n(n^*-s) + x/\gamma)^2}{2(n^*-s)}}\,.
\end{aligned}$$

Substituting the preceding inequality and (77) into (76), it follows that, for an absolute constant $C > 0$,

$$\mathbb{P}(E_{v,N}(z) \cap E_{w,N}(z)) \lesssim \mathrm{e}^{Cz^{1/10}} 4^{-n^*-s} \sum_{y \leqslant z} \mathrm{e}^{-\alpha_n y/\gamma} \gamma_{v,y}(z-y) \sum_{y \leqslant z} \mathrm{e}^{-\alpha_n y/\gamma} \gamma_{w,y}(z-y)\,.$$

Recalling that $\alpha_n/\gamma = \sqrt{2\pi} + O(\log n)/n$, an application of Lemma 4.9 therefore yields

$$\begin{aligned}
\mathbb{P}(E_{v,N}(z) \cap E_{w,N}(z)) \lesssim& \frac{\mathrm{e}^{2Cz^{1/10}}}{4^{n^*+s}\mathrm{e}^{2\sqrt{2\pi}z}} \sum_{y_1, y_2 \leqslant z} \mathrm{e}^{O(\frac{\log n (y_1+y_2)}{n})} \frac{(z-y_1)^2(z-y_2)^2}{\ell^3 \mathrm{e}^{((z-y_1)^2 + (z-y_2)^2)/C\ell}} \\
\lesssim& \mathrm{e}^{2Cz^{1/10}} 4^{-n^*-s} \mathrm{e}^{-2\sqrt{2\pi}z}\,.
\end{aligned}$$

Consequently,

$$\sum_{n^*-z^{1/10}\leqslant s\leqslant n^*}\sum_{v\sim_s w}\mathbb{P}(E_{v,N}(z)\cap E_{w,N}(z))\lesssim \mathrm{e}^{2Cz^{1/10}}\mathrm{e}^{-\sqrt{2\pi}z}\mathbb{E}\Lambda_{N,z}\,. \tag{78}$$

We next consider the case $z^{1/10}\leqslant s<n^*-z^{1/10}$. Here, (77) still holds (since the distance between v and w is large enough such that they belong to different boxes in $\mathcal{B}_N$). By the ballot theorem together with the change of measure that transforms $\bar{X}_{v,N}(\cdot)$ into Brownian motion,

$$\mathbb{P}(\bar{X}_{v,N}(t)\leqslant z,\ \text{for all } t\in[0,t_s], \bar{X}_{v,N}(t_s)\in[x-1,x])\lesssim \frac{z(z-x+1)}{(n^*-s)^{3/2}}\mathrm{e}^{-\frac{\alpha_n^2(n^*-s)}{2}}\mathrm{e}^{-\frac{\alpha_n x}{\gamma}}\,. \tag{79}$$

Substitution of (77) and the above estimate into (76) implies that

$$\begin{aligned}
&\mathbb{P}(E_{v,N}(z)\cap E_{w,N}(z))\\
\lesssim&\frac{z(n^*)^{3/2}}{4^{n^*+s}s^{3/2}(n^*-s)^{3/2}}\sum_{x\leqslant z}\mathrm{e}^{\alpha_n x/\gamma}(z-x)^3\sum_{y_1,y_2\leqslant z}\mathrm{e}^{-\alpha_n(y_1+y_2)/\gamma}(z-y_1)(z-y_2)\gamma_{v,y_1,w,y_2}\\
\lesssim&\frac{z(n^*)^{3/2}\mathrm{e}^{\sqrt{2\pi}z}}{4^{n^*+s}s^{3/2}(n^*-s)^{3/2}}\sum_{y_1,y_2\leqslant z}\mathrm{e}^{-\alpha_n(y_1+y_2)/\gamma}(z-y_1)(z-y_2)\gamma_{v,y_1}\gamma_{w,y_2}\,.
\end{aligned}$$

Combining this with Lemma 4.9, it follows that

$$\begin{aligned}
\mathbb{P}(E_{v,N}(z)\cap E_{w,N}(z))\lesssim&\frac{(\log z)^C z(n^*)^{3/2}\mathrm{e}^{-\sqrt{2\pi}z}}{4^{n^*+s}s^{3/2}(n^*-s)^{3/2}}\sum_{y_1,y_2\leqslant z}\mathrm{e}^{O(\frac{\log n(y_1+y_2)}{n})}\frac{(z-y_1)^2(z-y_2)^2}{\ell^3\mathrm{e}^{((z-y_1)^2+(z-y_2)^2)/C\ell}}\\
\lesssim&\frac{(\log z)^C z(n^*)^{3/2}\mathrm{e}^{-\sqrt{2\pi}z}}{4^{n^*+s}s^{3/2}(n^*-s)^{3/2}}\,.
\end{aligned}$$

Therefore,

$$\sum_{z^{1/10}\leqslant s<n-z^{1/10}}\sum_{v\sim_s w}\mathbb{P}(E_{v,N}(z)\cap E_{w,N}(z))\lesssim(\log z)^C z^{-1/20}z\mathrm{e}^{-\sqrt{2\pi}z}\lesssim(\log z)^C z^{-1/20}\mathbb{E}\Lambda_{N,z}\,. \tag{80}$$

Lastly, we consider the case $1\leqslant s<z^{1/10}$. Let $v\sim_s w$, with $v\neq w$, and define $\Gamma_{v,w,x,z,s}$ by

$$\sup_{\bar{X}_{v,N}(t_s)\in[x-1,x]}\mathbb{P}(\bar{X}_{v,N}(t),\bar{X}_{w,N}(t)\leqslant z \text{ for all } t_s<t\leqslant n^*; \max_{u\in\tilde{B}_v}\eta_{u,N}, \max_{u\in\tilde{B}_w}\eta_{u,N}\geqslant m_N+z\mid \bar{X}_{v,N}(t_s)).$$

Analogous to (76),

$$\mathbb{P}(E_{v,N}(z)\cap E_{w,N}(z))\leqslant\sum_{x\leqslant z}\mathbb{P}(\bar{X}_{v,N}(t)\leqslant z,\ \text{for all } t\in[0,t_s],\bar{X}_{v,N}(t_s)\in[x-1,x])\Gamma_{v,w,x,z,s}\,. \tag{81}$$

Furthermore,

$$\begin{aligned}
\Gamma_{v,w,x,z,s}\leqslant&\sum_{y_1,y_2\leqslant z}\mathbb{P}(\bar{X}_{v,N}(r),\bar{X}_{w,N}(r)\leqslant z-x \text{ for all } r\in[0,s];\\
&\qquad\bar{X}_{v,N}(s)\in[y_1-1-x,y_1-x];\bar{X}_{w,N}(s)\in[y_2-x-1,y_2-x])\gamma_{v,y_1,w,y_2}\\
\lesssim&\sum_{y_1,y_2\leqslant z}\mathrm{e}^{-\alpha_n^2 s}\mathrm{e}^{-\alpha_n(y_1+y_2-2x)/\gamma}s^{-1}\mathrm{e}^{-\frac{(y_1-x)^2+(y_2-x)^2}{2\gamma^2 s}}\gamma_{v,y_1,w,y_2}\,,
\end{aligned}$$

where $\gamma_{v,y_1,w,y_2} := \mathbb{P}(\max_{u\in\tilde{B}_v} Y_{u,N} \geqslant \ell m_N/N+z-y_1, \max_{u\in\tilde{B}_w} Y_{u,N} \geqslant \ell m_N/N+z-y_2)$. Together with Lemma 4.9, the last display implies

$$\begin{aligned}\Gamma_{v,w,x,z,s} &\lesssim (\log z)^C \mathrm{e}^{-\alpha_n^2 s}\mathrm{e}^{2\alpha_n x/\gamma}\mathrm{e}^{-2\sqrt{2\pi}z}\sum_{y_1,y_2\leqslant z}\frac{\mathrm{e}^{O(\log n(y_1+y_2)/n)}}{\mathrm{e}^{((y_1-x)^2+(y_2-x)^2)/2\gamma^2 s}}\frac{(z-y_1)(z-y_2)}{\ell^3 s}\\ &\lesssim (\log z)^C \mathrm{e}^{-\alpha_n^2 s}\mathrm{e}^{2\alpha_n x/\gamma}\mathrm{e}^{-2\sqrt{2\pi}z}(z-x+\sqrt{s}+\log\ell)^2\ell^{-3}\,.\end{aligned}$$

Note that (79) also holds in this region. Plugging (79) and the above inequality into (81), we obtain

$$\begin{aligned}\mathbb{P}(E_{v,N}(z)\cap E_{w,N}(z)) &\lesssim 4^{-n^*-s}s^{3/2}(\log z)^C\ell^{-3}z\mathrm{e}^{-\sqrt{2\pi}z}\sum_{x\leqslant z}(z-x+\sqrt{s}+\log\ell)^3\mathrm{e}^{-\alpha_n(z-x)/\gamma}\\ &\lesssim 4^{-n^*-s}s^{3/2}(\sqrt{s}+\log\ell)^3(\log z)^C\ell^{-3}z\mathrm{e}^{-\sqrt{2\pi}z}\,.\end{aligned}$$

Therefore, since $\ell \geqslant z^4$,

$$\sum_{1\leqslant s<z^{1/10}}\sum_{v\sim_s w, v\neq w}\mathbb{P}(E_{v,N}(z)\cap E_{w,N}(z)) \lesssim (\log z)^C\ell^{3/10}\ell^{-3}z\mathrm{e}^{-\sqrt{2\pi}z} \lesssim (\log z)^C\ell^{-2}\mathbb{E}\Lambda_{N,z}\,. \quad (82)$$

In each of the inequalities (78), (80) and (82), the coefficient of $\mathbb{E}\Lambda_{N,z}$ on the right hand side goes to 0 as $z\to\infty$. This shows that the sum in (75) is small in comparison with the preceding term for large z, and hence completes the proof of the lemma. □

Proof of Proposition 4.8. By an almost identical argument to that in Lemma 3.7, we obtain that

$$\mathbb{P}(G_N^{\mathrm{up}}(z)) \lesssim \mathrm{e}^{-\sqrt{2\pi}z}\,.$$

(The difference between the MBRW and the BRW does not affect this estimate; neither does the modification of the MBRW used for the process $\{X_{v,N}^{\mathrm{up}}\}$. Note that the slackness factor $z^{1/20}$ has been employed to kill the prefactor z in Lemma 3.7 (compare the definition (60) with (28)).) Combining this inequality with Lemma 4.10 and the trivial estimate

$$\mathbb{P}(G_N^{\mathrm{up}}(z)) + \mathbb{E}\Gamma_{N,z}^{\mathrm{up}} \geqslant \mathbb{P}(\max_{v\in\tilde{V}_N}\eta_{v,N}^{\mathrm{up}} \geqslant m_N+z)\,,$$

the upper bound

$$\limsup_{z\to\infty}\limsup_{N\to\infty}\frac{\mathbb{P}(\max_{v\in\tilde{V}_N}\eta_{v,N}^{\mathrm{up}}\geqslant m_N+z)}{\mathbb{E}\Lambda_{N,z}^{\mathrm{up}}} \leqslant 1$$

follows. The lower bound

$$\liminf_{z\to\infty}\liminf_{N\to\infty}\frac{\mathbb{P}(\max_{v\in\tilde{V}_N}\eta_{v,N}^{\mathrm{lw}}\geqslant m_N+z)}{\mathbb{E}\Lambda_{N,z}^{\mathrm{lw}}} \geqslant 1$$

follows from Lemma 4.11 and

$$\mathbb{P}(\max_{v\in\tilde{V}_N}\eta_{v,N}^{\mathrm{lw}}\geqslant m_N+z) \geqslant \mathbb{P}(\bigcup_{v\in\Xi_N}E_{v,N}^{\mathrm{lw}}(z)) \geqslant \frac{(\mathbb{E}\Lambda_{N,z}^{\mathrm{lw}})^2}{\mathbb{E}(\Lambda_{N,z}^{\mathrm{lw}})^2}\,.$$

The other statements follow from (58) and (61). □

For future reference, we note here that the same proof as for Lemma 4.10 also implies that, for any box $A\subset[\delta,1-\delta]^2$,

$$|A|ze^{-\sqrt{2\pi}z} \lesssim \mathbb{E}\Lambda_{N,z}^{\mathrm{lw}}(A)\,, \quad (83)$$

where $|A|$ denotes the area of A.

4.3 Asymptotics for the enumeration of large clusters and completion of the proof of Proposition 4.3.

This subsection is devoted to demonstrating Proposition 4.12, which gives the asymptotic behavior of $E\Lambda^{\mathrm{lw}}_{N,z}$ for large N and z.

Proposition 4.12. *There exists a constant $\alpha^*_\delta > 0$ such that*

$$\lim_{z\to\infty}\limsup_{N\to\infty}\frac{\mathbb{E}\Lambda^{\mathrm{lw}}_{N,z}}{\alpha^*_\delta z \mathrm{e}^{-\sqrt{2\pi}z}} = \lim_{z\to\infty}\liminf_{N\to\infty}\frac{\mathbb{E}\Lambda^{\mathrm{lw}}_{N,z}}{\alpha^*_\delta z \mathrm{e}^{-\sqrt{2\pi}z}} = 1\,.$$

Furthermore, there exist continuous functions $\psi_\delta : [\delta, 1-\delta]^2 \mapsto (0,\infty)$, with $\int_{[\delta,1-\delta]^2}\psi_\delta(x)dx = 1$, and a continuous function $\psi : (0,1)^2 \mapsto (0,\infty)$ such that $\psi_\delta(x) \to \psi(x)$ uniformly in x on closed sets, as $\delta \searrow 0$, and such that, for any open box $A \subseteq [\delta, 1-\delta]^2$,

$$\lim_{z\to\infty}\limsup_{N\to\infty}\frac{\mathbb{E}\Lambda^{\mathrm{lw}}_{N,z}(A)}{\alpha^*_\delta z \mathrm{e}^{-\sqrt{2\pi}z}} = \lim_{z\to\infty}\liminf_{N\to\infty}\frac{\mathbb{E}\Lambda^{\mathrm{lw}}_{N,z}(A)}{\alpha^*_\delta z \mathrm{e}^{-\sqrt{2\pi}z}} = \int_A \psi_\delta(x)dx\,.$$

Together, Propositions 4.7 and 4.12 imply Proposition 4.3 for open boxes $A \subseteq [\delta, 1-\delta]^2$, which easily extends to open sets in $[\delta, 1-\delta]^2$.

To simplify notation, we drop the lw superscript in the rest of the subsection. For $v \in \Xi_N$, let $\nu_{v,N}(\cdot)$ be the density function (of a probability measure on $\mathbb{R}$) such that, for all $I \subseteq \mathbb{R}$,

$$\int_I \nu_{v,N}(y)dy = \mathbb{P}(X_{v,N}(t) \leqslant z + \frac{m_N}{n}t \text{ for all } 0 \leqslant t \leqslant n_{v,N}; X_{v,N}(n_{v,N}) - (n-\ell)m_N/n \in I)\,.$$

Clearly,

$$\mathbb{P}(E_{v,N}(z)) = \int_{-\infty}^{z} \nu_{v,N}(y)\mathbb{P}(\max_{u\in\tilde{B}_v} Y_{u,N} \geqslant \ell m_N/n + z - y)dy\,.$$

Recall the variables $\ell, \tilde{\ell}$ defined at the beginning of the section and, for a given interval J, define

$$\lambda_{v,N,z,J} = \int_J \nu_{v,N}(y)\mathbb{P}(\max_{u\in\tilde{B}_v} Y_{u,N} \geqslant \ell m_N/n + z - y)dy\,. \tag{84}$$

Set $J_\ell = [-\ell, -\ell^{2/5}]$. The following estimate shows that the main contribution to $\mathbb{E}\Lambda_{N,z}(A)$ is from values $y \in J_\ell$, as in (84). (The choice of the exponent $2/5$ here is somewhat arbitrary; only $0 < 2/5 < 1/2$ is used.)

Lemma 4.13. *For any box $A \subseteq [\delta, 1-\delta]^2$ and any sequence $x_{v,N}$ such that $x_{v,N} \lesssim \ell^{2/5}$,*

$$\lim_{z\to\infty}\liminf_{N\to\infty}\frac{\sum_{v\in\Xi_N\cap NA}\lambda_{v,N,z,x_{v,N}+J_\ell}}{\mathbb{E}\Lambda_{N,z}(A)} = 1\,.$$

Proof. We prove the lemma for the case when $x_{v,N} = 0$; the general case follows in the same manner. Application of the reflection principle (23) to the Brownian motion with drift, $\bar{X}_{v,N}(\cdot) = X_{v,N}(\cdot) - m_N t/n$, together with the change of measure that removes the drift $m_N t/n$, implies that

$$\nu_{v,N}(y) \lesssim e^{-\sqrt{2\pi}y}4^{-n_{v,N}}z\ell$$

for $y \leqslant -\ell$, over the given range $z \in (0, \ell)$. Together with Lemma 4.9, this implies the crude bound

$$\int_{-\infty}^{-\ell} \nu_{v,N}(y) \mathbb{P}(\max_{u \in \tilde{B}_{v,N}} Y_{u,N} \geqslant \ell m_N/n + z - y) dy \lesssim 4^{-n_{v,N}} \mathrm{e}^{-C^{-1}\ell}$$

for an absolute constant $C > 0$. Similarly, for $y \leqslant z$ (and therefore, for $z - y \geqslant 0$), application of the reflection principle and Lemma 4.9 again implies that

$$\int_{-\ell^{2/5}}^{z} \nu_{v,N}(y) \mathbb{P}(\max_{u \in \tilde{B}_{v,N}} Y_{u,N} \geqslant \ell m_N/n + z - y) dy \lesssim 4^{-n_{v,N}} \ell^{-3/10} (\log z)^C z \mathrm{e}^{-\sqrt{2\pi} z} .$$

Together with (83), this implies that $\mathbb{E}\Lambda_{N,z}(A) - \sum_{v \in \Xi_N \cap NA} \lambda_{v,N,z,J_\ell} \lesssim \ell^{-3/10} (\log z)^C \mathbb{E}\Lambda_{N,z}(A)$, as needed. $\square$

Lemma 4.14. *There exists Λ_z^* depending only on z such that, for all functions L and $\tilde{L}$ of z satisfying* (40),

$$\lim_{z\to\infty} \limsup_{N\to\infty} \frac{\mathbb{E}\Lambda_{N,z}}{\Lambda_z^*} = \lim_{z\to\infty} \liminf_{N\to\infty} \frac{\mathbb{E}\Lambda_{N,z}}{\Lambda_z^*} = 1 .$$

Furthermore, there exist continuous functions $\psi_\delta : [\delta, 1-\delta]^2 \mapsto (0, \infty)$, with $\int_{[\delta,1-\delta]^2} \psi_\delta(x) dx = 1$, and a continuous function $\psi : (0,1)^2 \mapsto (0, \infty)$ such that $\psi_\delta \to \psi$ uniformly in x on closed subsets of $(0,1)^2$, as $\delta \searrow 0$, and such that, for any box $A \subseteq [\delta, 1-\delta]^2$,

$$\lim_{z\to\infty} \limsup_{N\to\infty} \frac{\mathbb{E}\Lambda_{N,z}(A)}{\Lambda_z^*} = \lim_{z\to\infty} \liminf_{N\to\infty} \frac{\mathbb{E}\Lambda_{N,z}(A)}{\Lambda_z^*} = \int_A \psi_\delta(x) dx .$$

Proof. By Proposition 4.8, if the claim holds for a particular choice of $\tilde{L}(z) \geqslant 2^{z^4}$, then it must hold for all choices. So, it suffices to consider the case when $\tilde{L}(z) = 2^{z^4}$.

Write $x_{v,N} = m_N(1 - n_{v,N}/n) - \gamma\sqrt{2\pi}\ell$. It follows from (1), (55) and (56) that $x_{v,N} = O(1)$. (Recall that $\gamma = \sqrt{2\log 2/\pi}$ and $\mathrm{Var}(S^{\mathrm{lw}}_{v,N,r}) = \gamma^2(n - \ell - r + 1)$.) For $\hat{v} \in [-L/2, L/2]^2$, set $\Xi_{\hat{v},N} = \{v \in \Xi_N : v - c_{B_v} = \hat{v}\}$, where c_{B_v} is the center of the box $B_v \in \mathcal{B}_N$ containing v. Define

$$\Lambda_{\hat{v},N,z,J_\ell} = \sum_{v \in \Xi_{\hat{v},N}} \int_{J_\ell - x_{v,N}} \nu_{v,N}(y) \mathbb{P}(\max_{u \in \tilde{B}_v} Y_{u,N} \geqslant \ell m_N/n + z - y) dy ,$$

$$\Lambda_{\hat{v},N,z,J_\ell}(A) = \sum_{v \in \Xi_{\hat{v},N} \cap AN} \int_{J_\ell - x_{v,N}} \nu_{v,N}(y) \mathbb{P}(\max_{u \in \tilde{B}_v} Y_{u,N} \geqslant \ell m_N/n + z - y) dy ,$$

where J_ℓ is as in Lemma 4.13. Note that, except for about $h^2 = (L/\tilde{L})^2$ values of $\hat{v}$, $\Xi_{\hat{v},N} = \emptyset$. In light of Lemma 4.13, it suffices to show, for an arbitrary $\hat{v} \in [-L/2, L/2]^2$ with $\Xi_{\hat{v},N} \neq \emptyset$, that there exists $\Lambda^*_{\hat{v},z}$ satisfying

$$\limsup_{N\to\infty} \frac{\mathbb{E}\Lambda_{\hat{v},N,z,J_\ell}}{\Lambda^*_{\hat{v},z}} = 1 + O(z^{-1}) = \liminf_{N\to\infty} \frac{\mathbb{E}\Lambda_{\hat{v},N,z,J_\ell}}{\Lambda^*_{\hat{v},z}} , \tag{85}$$

$$\limsup_{N\to\infty} \frac{\mathbb{E}\Lambda_{\hat{v},N,z,J_\ell}(A)}{\Lambda^*_{\hat{v},z}} = (1 + O(z^{-1})) \int_A \psi_\delta(x) dx = \liminf_{N\to\infty} \frac{\mathbb{E}\Lambda_{\hat{v},N,z,J_\ell}(A)}{\Lambda^*_{\hat{v},z}} , \tag{86}$$

where $\psi_\delta : [\delta, 1-\delta]^2 \mapsto (0, \infty)$ is a continuous function with $\int_{[\delta,1-\delta]^2} \psi_\delta(x) = 1$, and ψ_δ converges to a continuous function as $\delta \searrow 0$. (The rates of convergence will not depend on the choice of $\hat{v}$.)

Note that, crucially, the function ψ_δ is required to be *independent* of the choice of $(\hat{v}, z, L, \tilde{L})$. It is clear that, for all $v \in \Xi_{\hat{v},N}$, the distribution of $M_{\hat{v},z} := \max_{u \in \tilde{B}_v} Y_{u,N}$ depends only on $\hat{v}$, L, $\tilde{L}$ and z.

By (70) and the reflection principle,

$$\nu_{v,N}(y + x_{v,N}) = 4^{-n_{v,N}} \mathrm{e}^{-\sqrt{2\pi}(y+x_{v,N})} \frac{z(2z - y - x_{v,N})}{\sqrt{2\pi}\gamma} (1 + O(\ell^3/n)) \mathbf{1}_{\{x_{v,N}+y \leqslant z\}} . \tag{87}$$

Therefore,

$$\begin{aligned} \Lambda_{\hat{v},N,z,J_\ell} &= \sum_{v \in \Xi_{\hat{v},N}} \int_{J_\ell} \nu_{v,N}(y + x_{v,N}) \mathbb{P}(M_{\hat{v},z} \geqslant \sqrt{2\pi}\gamma\ell + z - y) dy \\ &= \int_{J_\ell} \sum_{v \in \Xi_{\hat{v},N}} 4^{-n_{v,N}} \frac{z(2z - y - x_{v,N})}{\sqrt{2\pi}\gamma \mathrm{e}^{\sqrt{2\pi}(y+x_{v,N})}} \mathbb{P}(M_{\hat{v},z} \geqslant \sqrt{2\pi}\gamma\ell + z - y) dy + O(\ell^3/n) \end{aligned}$$

and

$$\Lambda_{\hat{v},N,z,J_\ell}(A) = \int_{J_\ell} \sum_{v \in \Xi_{\hat{v},N} \cap NA} 4^{-n_{v,N}} \frac{z(2z - y - x_{v,N})}{\sqrt{2\pi}\gamma \mathrm{e}^{\sqrt{2\pi}(y+x_{v,N})}} \mathbb{P}(M_{\hat{v},z} \geqslant \sqrt{2\pi}\gamma\ell + z - y) dy + O(\ell^3/n) .$$

Note that $x_{v,N}$ is a linear continuous function and $n_{v,N}$ is a quadratic continuous function of $a_{v,N}$ (as defined in (55)). For any point v^* in the unit square, denote by v^*_N the vertex in $\Xi_{\hat{v},N}$ that is closest to Nv^*. By (57), for any $v^* \in [\delta, 1-\delta]^2$, the limit

$$\phi_{\hat{v}}(v^*) = \lim_{N \to \infty} 4^{n-\ell} 4^{-n_{v^*_N,N}} \mathrm{e}^{-\sqrt{2\pi} x_{v^*_N,N}}$$

exists and $\phi_{\hat{v}}(v^*)$ is a continuous function on $[0,1]^2$ since $a_{v,N}$ varies smoothly in $v \in \Xi_{\hat{v},N}$. Furthermore, it follows from the definitions of $a_{v,N}$, $n_{v,N}$ and $x_{v,N}$ that

$$\frac{\phi_{\hat{v}}(v^*)}{\phi_{\hat{v}}(u^*)} \text{ is a function depending only on } (v^*, u^*) . \tag{88}$$

By the bounded convergence theorem,

$$\limsup_{N \to \infty} \Lambda_{\hat{v},N,z,J_\ell} = (1 + O(z^{-1})) \int_{J_\ell} \frac{z(2z - y)}{\sqrt{2\pi}\gamma \mathrm{e}^{\sqrt{2\pi} y}} \mathbb{P}(M_{\hat{v},z} \geqslant \sqrt{2\pi}\gamma\ell + z - y) dy \cdot h^2 \int_{[\delta,1-\delta]^2} \phi_{\hat{v}}(x) dx ,$$

with a similar estimate holding when lim sup is replaced by lim inf. Since $\mathbb{P}(M_{\hat{v},z} \geqslant \sqrt{2\pi}\gamma\ell + z - y)$ is a function of just $(\hat{v}, z, y)$, this completes the proof of (85). Similarly,

$$\limsup_{N \to \infty} \Lambda_{\hat{v},N,z,J_\ell}(A) = (1 + O(z^{-1})) \int_{J_\ell} \frac{z(2z - y)}{\sqrt{2\pi}\gamma \mathrm{e}^{\sqrt{2\pi} y}} \mathbb{P}(M_{\hat{v},z} \geqslant \sqrt{2\pi}\gamma\ell + z - y) dy \cdot h^2 \int_A \phi_{\hat{v}}(x) dx ,$$

with a similar estimate holding when lim sup is replaced by lim inf. Setting

$$\psi_{\delta,\hat{v}}(x) = \phi_{\hat{v}}(x) / \int_{[\delta,1-\delta]^2} \phi_{\hat{v}}(x) dx ,$$

it follows that $\psi_{\delta,\hat{v}}$ satisfies all of the desired properties. In particular, by (88), the function $\psi_{\delta,\hat{v}}$ is independent of $(\hat{v}, z, L, \tilde{L})$. This completes the proof of (86) and hence the proof of the lemma. □

We are now ready to prove Proposition 4.12.

Proof of Proposition 4.12. The second display in Proposition 4.12 follows directly from the first display and the second display in Lemma 4.14. It therefore suffices to prove the first display in Proposition 4.12. To this end, consider $z_1 < z_2$, and set $\tilde{L} = 2^{z_2^4}$ and $h = \log z_1$. For $v \in \Xi_N$ and $i = 1, 2$, set

$$\lambda_{v,N,z_i,z_i+J_\ell} = \int_{J_\ell + z_i} \nu_{v,N}(y)\mathbb{P}(\max_{u \in \tilde{B}_{v,N}} Y_{u,N} \geqslant \ell m_N/n + z_i - y)dy\,.$$

By (87), for any $y \in J_\ell$,

$$\begin{aligned}&\frac{\nu_{v,N}(y+z_1)\mathbb{P}(\max_{u\in\tilde{B}_{v,N}} Y_{u,N} \geqslant \ell m_N/n - y)}{\nu_{v,N}(y+z_2)\mathbb{P}(\max_{u\in\tilde{B}_{v,N}} Y_{u,N} \geqslant \ell m_N/n - y)}\\ &= \frac{\nu_{v,N}(y+z_1)}{\nu_{v,N}(y+z_2)} = \frac{z_1(z_1-y)}{z_2(z_2-y)}\mathrm{e}^{-\sqrt{2\pi}(z_1-z_2)} + O(\ell^3/n) = \frac{z_1}{z_2}\mathrm{e}^{-\sqrt{2\pi}(z_1-z_2)}(1+z_2^{-3/5}) + O(\ell^3/n)\,.\end{aligned}$$

This implies that

$$\frac{\lambda_{v,N,z_1,z_1+J_\ell}}{\lambda_{v,N,z_2,z_2+J_\ell}} = \frac{z_1}{z_2}\mathrm{e}^{-\sqrt{2\pi}(z_1-z_2)}(1+z_2^{-3/5}) + O(\ell^3/n)\,.$$

Together with Lemma 4.13, the above display implies that

$$\lim_{z_1,z_2\to\infty}\limsup_{N\to\infty}\frac{\mathbb{E}\Lambda_{N,z_1}}{\mathbb{E}\Lambda_{N,z_2}} = \lim_{z_1,z_2\to\infty}\liminf_{N\to\infty}\frac{\mathbb{E}\Lambda_{N,z_1}}{\Lambda_{\mathbb{E}N,z_2}} = \frac{z_1}{z_2}\mathrm{e}^{-\sqrt{2\pi}(z_1-z_2)}\,.$$

Along with Lemma 4.14, this completes the proof of the proposition. □

5 A pair of approximations

The main results in this section are Propositions 5.1 and 5.2. Proposition 5.1 will be applied in Section 6, and allows us to restrict our attention to the sets $V_N^{K,\delta} = \cup_i V_N^{K,\delta,i}$ when computing the maximum of $\eta_{v,N}$.

Proposition 5.1. *With notation as defined earlier,*

$$\limsup_{\delta\searrow 0}\limsup_{K\to\infty}\limsup_{N\to\infty}\mathbb{P}(\max_{v\in V_N^{K,\delta}}\eta_{v,N} \neq \eta_N^*) = 0. \tag{89}$$

Proof. Due to the tightness of the sequence of random variables $(\eta_N^* - m_N)$ (see [5]), it suffices to show that, for any fixed $x \in \mathbb{R}$,

$$\limsup_{\delta\searrow 0}\limsup_{K\to\infty}\limsup_{N\to\infty}\mathbb{P}(\max_{v\in\Delta_N}\eta_{v,N} - m_N \geqslant x) = 0\,. \tag{90}$$

The claim (90) follows at once from (34). □

Proposition 5.2 will be applied in conjunction with Proposition 5.1, and implies that the local maxima of the GFF occur at the local maxima of the fine field, at least when restricted to $V_K^{N,\delta}$.

Proposition 5.2. *Let $z_i = z_i^{N,K,\delta}$ be such that*

$$\max_{v\in V_N^{K,\delta,i}} X_v^f = X_{z_i}^f\,.$$

Let $\bar z = \bar z(N,K,\delta)$ be such that

$$\max_i \eta_{z_i,N} = \eta_{\bar z,N}\,.$$

Then, for any fixed $\varepsilon, \delta > 0$,

$$\limsup_{K\to\infty}\limsup_{N\to\infty} \mathbb{P}(\max_{v\in V_N^{K,\delta}} \eta_{v,N} \geqslant \eta_{\bar z,N} + \varepsilon) = 0\,. \tag{91}$$

Furthermore, there exists a function $g : \mathbb{N} \to \mathbb{R}_+$, with $g(K) \to_{K\to\infty} \infty$, such that

$$\limsup_{K\to\infty}\limsup_{N\to\infty} \mathbb{P}(X_{\bar z}^f \leqslant m_{N/K} + g(K)) = 0\,. \tag{92}$$

The proof of Proposition 5.2 occupies the remainder of the section.

Proof. The strategy for the proof of (91) is as follows. Suppose the event on the right hand side of (91) occurs. This means, in particular, that either in one of the boxes $V_N^{K,\delta,i}$ the difference between $\eta_{v_i} := \max_{v\in V_N^{K,\delta,i}} \eta_{v,N}$ and η_{z_i} is large, or that two near-maxima of the GFF lie within distance N/K of one another. By properties of Gaussian fields (in particular, the uniform continuity of X_v^c), the former event is unlikely. By [8], the latter event is possible only if the distance between v_i and z_i is of order 1, not depending on K. However, the continuity of the field of X^c implies that, within such distances, the difference $|\eta_{v_i,N} - \eta_{z_i,N}|$ will be small.

Turning to the actual proof of (91), fix two constants $C, C' > 0$. Suppose that the event on the left hand side of (91) occurs and that $\max_{v\in V_N^{K,\delta}} \eta_{v,N} \geqslant m_N - C$. Recall that $k = \log_2 K$, and fix a function $f : \mathbb{N} \to \mathbb{R}_+$, with $f(k) \to_{k\to\infty} \infty$. Then (keeping in mind the above description), one of the following events must occur:

- $\mathcal{A}_1 := \{\max_{v\in V_N^{K,\delta}} \max_{w\in V_N^{K,\delta}:|v-w|\leqslant f(k)} |X_v^c - X_w^c| \geqslant \varepsilon\}\,,$
- $\mathcal{A}_2 := \{\max_i \max_{u,v\in V_N^{K,\delta,i}} (\eta_{u,N} + X_v^c - X_u^c) \geqslant m_N + C'\}\,,$
- $\mathcal{A}_3 := \{\exists i, v : N/K \geqslant d(v, z_i) > f(k),\, \eta_{v,N} \geqslant m_N - C, \eta_{z_i,N} \geqslant m_N - C - C'\}.$

We will show that the probability of each of these three events is small.

By a union bound and Lemma 3.10, with $\mathcal{N}$ denoting a standard Gaussian random variable,

$$\mathbb{P}(\mathcal{A}_1) \leqslant N^2 f(k)^2 \mathbb{P}(\sqrt{Kc_\delta f(k)/N}\mathcal{N}) > \varepsilon) \leqslant N^2 f(k)^2 e^{-\varepsilon^2 N/2Kc_\delta f(k)} \to_{N\to\infty} 0\,. \tag{93}$$

On the other hand, by [8, Theorem 1.1], for any fixed C, C',

$$\limsup_{K\to\infty}\limsup_{N\to\infty} \mathbb{P}(\mathcal{A}_3) = 0\,.$$

This limit employs $f(k) \to_{k\to\infty} \infty$.

We will show below that, for some constant C_δ not depending on N, K, C,

$$\mathbb{P}(\mathcal{A}_2) \leqslant \frac{C_\delta}{C' - C_\delta} \leqslant \frac{2C_\delta}{C'}\,, \tag{94}$$

for large enough C'. Taking $N \to \infty$, followed by $K \to \infty$, then $C \to \infty$ and then $C' \to \infty$, implies $P(\mathcal{A}_2) \to 0$. The above limits on $P(\mathcal{A}_i)$, $i = 1, 2, 3$, together imply (91).

In order to estimate $\mathbb{P}(\mathcal{A}_2)$, we require a couple of lemmas. Write $Y_{u,v} = \eta_{u,N} + X_v^c - X_u^c$, and recall that u, v belong to the same box $V_K^{N,\delta,i}$. Write $V_{N,K,\delta}^{\times 2} := \{(u, v) : u, v \in V_K^{N,\delta,i} \text{ for some } i\}$. The proof of the first lemma is a straightforward application of Lemma 3.10, and is therefore omitted.

Lemma 5.3. *There exists a constant c_1 independent of K, N such that, for $(u, v), (u', v') \in V_{N,K,\delta}^{\times 2}$,*

$$\mathbb{E}(Y_{u,v} - Y_{u',v'})^2 \leqslant \mathbb{E}(\eta_u - \eta_v)^2 + c_1 \frac{\max(|v - u|, |v' - u'|)}{N/K}\,. \tag{95}$$

We next construct a MBRW ξ_u in a box of size N, using only the top k levels. That is, with $\mathcal{B}_j^N$ denoting the collection of subsets of $\mathbb{Z}^2$ consisting of squares of side length 2^j with lower left corner in V_N, and with $\{b_{j,B}\}_{j \geqslant 0, B \in \mathcal{B}_j^N}$ denoting an i.i.d. family of centered Gaussian random variables of variance 2^{-2j}, independent of $\eta_{\cdot,N}$, let

$$\xi_{u,N} = \sum_{j=n-k}^{n} \sum_{B \in \mathcal{B}_j(u)} b_{j,B}^N\,.$$

(Here, $\mathcal{B}_j(u)$ denotes those elements of $\mathcal{B}_j^N$ that contain u.) Let $\{\xi_N^i\}_i$ denote an i.i.d. family of copies of ξ_N and, for $u, v \in V_{K,\delta,i}^N$, set $Z_{u,v}^N = \xi_{u,N}^i - \xi_{v,N}^i$. Thus, Z^N is a Gaussian field with index set $V_{N,K,\delta}^{\times 2}$.

Fix a constant $C_2 > 0$ and set $\bar{Y}_{u,v} = \eta_{u,N} + C_2 Z_{u,v}^N$. It follows immediately from Lemma 5.3 and a direct computation that there is a choice of C_2 such that, for any $(u, v), (u', v') \in V_{N,K,\delta}^{\times 2}$,

$$\mathbb{E}(\bar{Y}_{u,v} - \bar{Y}_{u',v'})^2 \geqslant \mathbb{E}(Y_{u,v} - Y_{u',v'})^2. \tag{96}$$

In particular, with $Y_N^* := \max_{(u,v) \in V_{N,K,\delta}^{\times 2}} Y_{u,v}$ and $\bar{Y}_N^* := \max_{(u,v) \in V_{N,K,\delta}^{\times 2}} \bar{Y}_{u,v}$, by lemma 3.2,

$$\mathbb{E} Y_N^* \leqslant \mathbb{E} \bar{Y}_N^*\,. \tag{97}$$

We make one more comparison. Let $\vartheta_{\cdot,N}$ be the 4-ary BRW indexed by V_N, chosen independently of $Z_{\cdot,\cdot}^N$, and set $\tilde{Y}_{u,v} = \vartheta_{u,N} + C_2 Z_{u,v}^N$ and $\tilde{Y}_N^* = \max_{(u,v) \in V_{N,K,\delta}^{\times 2}} \tilde{Y}_{u,v}$. By the domination of the correlation distance of GFF by that of the BRW (see [5]), Lemma 3.2 and (97), it follows that

$$\mathbb{E} Y_N^* \leqslant \mathbb{E} \bar{Y}_N^* \leqslant \mathbb{E} \tilde{Y}_N^*\,. \tag{98}$$

We need the following lemma, whose proof is postponed until later in this section.

Lemma 5.4. *There exists a constant $c_\delta > 0$, not depending on K, N, so that*

$$\mathbb{E} \tilde{Y}_N^* \leqslant m_N + c_\delta\,. \tag{99}$$

By Lemma 5.4 and (98), $\mathbb{E} Y_N^* \leqslant m_N + c_\delta$. On the other hand, by definition, $Y_N^* \geqslant X_N^* := \max_{v \in V_N^{K,\delta}} \eta_{v,N}$. Together with $\mathbb{E}(X_N^* - m_N)_- \leqslant C''$, which follows from the same argument as for

$\mathbb{E}(\eta_N^* - m_N)_- < C''$ (see [5, Pages 12–15]), this implies that $\mathbb{E}|Y_N^* - m_N| \leqslant C_\delta$ for some constant C_δ not depending on N, K. This demonstrates (94) and hence (91).

To demonstrate (92), fix $\varepsilon' > 0$ and, using Proposition 5.1, recall that

$$\liminf_{K\to\infty} \liminf_{N\to\infty} \mathbb{P}(\max_{z\in V_N^{K,\delta}} \eta_{z,N} \geqslant m_N - \varepsilon' \log k) = 1\,.$$

By (91), this implies

$$\liminf_{K\to\infty} \liminf_{N\to\infty} \mathbb{P}(\eta_{\bar{z},N} \geqslant m_N - \varepsilon' \log k) = 1\,.$$

Therefore, since $m_N - m_{N/K} = c^* k + C_N(K)$, with $c^* = 2\log 2 \cdot \sqrt{2/\pi}$ and $C_N(K) \to_{N\to\infty} 0$, for any fixed K, (92) will follow from

$$\limsup_{K\to\infty} \limsup_{N\to\infty} \mathbb{P}(\max_{i=1}^{K^2} X_{z_i}^c \geqslant c^* k - \varepsilon' \log k - g(K)) = 0\,.$$

But, for appropriate $g(\cdot)$, the latter is a consequence of a simple union bound: Setting $g(K) = \alpha \log k$, with $\alpha > 0$, and $\alpha' = \alpha + \varepsilon'$,

$$\mathbb{P}(\max_{i=1}^{K^2} X_{z_i}^c \geqslant c^* k - \alpha' \log k)) \leqslant \sum_{i=1}^{K^2} \mathbb{P}(X_{z_i}^c \geqslant c^* k - \alpha' \log k)\,.$$

Applying Lemma 3.1 together with the analog of (6), the mean zero normal $X_{z_i}^c$ has variance bounded above by $(c^*)^2 k/4\log 2 + c'$, for appropriate c'. So, the right hand side of the last display is bounded above by

$$CK^2 \frac{\mathrm{e}^{-2(\log 2)(c^* k - \alpha' \log k)^2/(c^*)^2 k}}{k^{1/2}} \leqslant C\mathrm{e}^{((4\alpha'(\log 2)/c^*) - 1/2)\log k}\,.$$

Choosing $\alpha \in (0, c^*/(8\log 2))$ and ε' small enough implies that the right hand side of this display $\to 0$ as $K \to \infty$, which completes the proof of (92). $\square$

We turn to the proof of Lemma 5.4.

Proof of Lemma 5.4. We begin by considering a modified field, where the variables $Z_{u,v}^N$ are replaced by variables $\bar{Z}_{u,v}^N$ so that, for $u \in V_N^{K,\delta}$, $\{\bar{Z}_{u,v}\}_{v\in B_{N/K,\delta}(u)}$ has the same law as $\{Z_{u,v}\}_{v\in B_{N/K,\delta}(u)}$, but the variables are independent for different u. Here, $B_{N/K,\delta}(u)$ denotes the box $V_N^{K,\delta,i}$ to which u belongs. Let $\tilde{Y}'_{u,v} = \vartheta_{u,N} + 2C_2 \bar{Z}_{u,v}^N$, with $\tilde{Y}_N^{',*} = \max_{(u,v)\in M_n} \tilde{Y}'_{u,v}$. Then $\mathbb{E}\tilde{Y}_N^* \leqslant \mathbb{E}\tilde{Y}_N^{',*}$ by Lemma 3.2.

For $u \in V_N^{K,\delta}$, set $\zeta_u = 2C_2 \max_{v\in B_{N/K,\delta}(u)} \bar{Z}_{u,v}^N$. Note that $\mathbb{E}|\bar{Z}_{u,v} - \bar{Z}_{u,v'}|^2 \leqslant C_\delta |v - v'|/(N/K)$. For $u \in V_N \setminus V_{NK,\delta}$, set $\zeta_u = 0$. A direct application of Fernique's criterion (Lemma 3.5, with the box B taken to be $B_{N/K,\delta}$) shows that $\mathbb{E}\zeta_u \leqslant c_0 = c_0(\delta)$. On the other hand, since for any u, v, $\mathbb{E}\bar{Z}_{u,v}^2 \leqslant c_1 = c_1(\delta)$, we conclude by Lemma 3.4 that

$$\mathbb{P}(\zeta_u \geqslant c_0 + y) \leqslant 2\mathrm{e}^{-y^2/2c_1}\,, \quad y \geqslant 0\,.$$

An application of Lemma 3.9 then completes the proof of Lemma 5.4. $\square$

6 Proof of Theorem 2.3

The proof of Theorem 2.3 is based on coupling the independent random variables (Y_i^K, z_i^K) of Subsection 2.3 with the values and locations of the local maxima of the fine field X^f. We begin with a construction of this coupling. We next prove a continuity property of the coarse field and then employ these two steps to demonstrate Theorem 2.3.

For probability measures ν_1, ν_2 on $\mathbb{R}$, we denote by $d(\nu_1, \nu_2)$ the Lévy distance between ν_1 and ν_2, i.e.,

$$d(\nu_1, \nu_2) = \min\{\delta > 0 : \nu_1(B) \leqslant \nu_2(B^\delta) + \delta \quad \text{for all open sets } B\},$$

where $B^\delta = \{y : |x - y| < \delta \text{ for some } x \in B\}$. With a slight abuse of notation, when X and Y are random variables with laws μ_X and μ_Y, respectively, we will also write $d(X, Y)$ for $d(\mu_X, \mu_Y)$.

6.1 The coupling construction

We begin with a preliminary lemma.

Lemma 6.1. *There exists a constant $C^* > 0$ so that, for all K,*

$$\limsup_{N\to\infty} \mathbb{P}(\max_{v\in V_{N/K}} \eta_{v,N/K} \geqslant m_{N/K} + C^* k) \leqslant K^{-3}\,. \tag{100}$$

Proof. Apply (11) with $A = (0,1)^2$. □

Let $g(K)$ be as in Proposition 5.2, and recall from the proof of the proposition that we can choose $g(K) = \alpha \log k$ for an appropriate $\alpha > 0$. Also, recall the variables Y_i^K in (12). Set $\theta_K(x) = \mathrm{e}^{\sqrt{2\pi}(x+g(K))}/(g(K)+x)$, for $x \geqslant 0$, and set $\eta^*_{N/K} = \max_{v\in V_{N/K}} \eta_{v,N/K}$.

Lemma 6.2. *There exist $\varepsilon_K \to_{K\to\infty} 0$ and a sequence of numbers $\alpha_{1,K,N} < \alpha_{2,K,N} < \ldots < \alpha_{C^*k,K,N}$ satisfying*

$$|\alpha_{j,K,N} - (g(K) + j - 1)| \leqslant \varepsilon_K \tag{101}$$

such that

$$\theta_K(0)^{-1}\mathbb{P}(Y_i^K \in [j-1, j]) = \mathbb{P}(\eta^*_{N/K} - m_{N/K} \in [\alpha_{j,K,N}, \alpha_{j+1,K,N}))\,,\ \textit{for } j = 1, \ldots, C^*k\,. \tag{102}$$

Proof. Setting $\beta = \beta_{N,K} = \theta_K(0)\mathbb{P}(\mathcal{A}_{N,K})$, it follows from (10) that, for all N large, $|\beta - \alpha^*| \leqslant \delta_K$, with $\delta_K \to_{K\to\infty} 0$. Using the uniform continuity of the function $1/\theta_K(\cdot)$ and (11), with $A = (0,1)^2$, the conclusion follows for an appropriate choice of ε_K. □

We now construct the required coupling. Recall the set $V_N^{K,\delta,1}$ defined in (3), with $V_N^{K,\delta,1} \subset V_{N/K}$ and consisting of points further than $\delta N/K$ from $\partial V_{N/K}$. Choose the enumeration of $\mathcal{W}_i$ in Section 2.3 so that $(N/K)\mathcal{W}_1 \cap \mathbb{Z}^2 = V_{N/K}$. Denote by $(\wp, Y, z^K)$ a copy of the random vector $(\wp_1^K, Y_1^K, z_1^K)$. Recall the random variable v^* defined by $\eta_{v^*,N/K} = \eta^*_{N/K}$.

Proposition 6.3. *There exists a sequence $\bar\varepsilon_K \to_{K\to\infty} 0$ such that $(\bar\wp_{N,K}, \eta^*_{N/K} - m_{N/K}, v^*)$ and $(\wp, Y, z^K)$ can be constructed on the same probability space, with $\wp = \bar\wp_{N,K} := 1_{\{\eta^*_{N/K} - m_{N/K} \geqslant \alpha_{1,K,N}\}}$ holding with probability 1, and such that, on the event where $v^* \in V_N^{K,\delta/2,1}$ and $\eta^*_{N/K} - m_{N/K} \leqslant \alpha_{C^*k,K,N}$,*

$$\wp|g(K) + Y - (\eta^*_{N/K} - m_{N/K})| + K|z^K - v^*/N| \leqslant \bar\varepsilon_K\,. \tag{103}$$

In words, Proposition 6.3 states that the two processes can be coupled so that, on the rare set where $\wp = 1$, $\eta^*_{N/K} - m_{N/K}$ is always closely approximated by $g(K) + Y$, and v^*/N is closely approximated by z^K.

Proof. Employing Lemma 6.2, there is a piecewise linear map $\ell : [\alpha_{1,K,N}, \alpha_{C^*k,K,N}) \mapsto [g(K), g(K)+C^*k)$, with $\ell(\alpha_{j,K,N}) = g(K) + j - 1$, for $j = 1, \ldots, C^*k$, and having Lipschitz constant contained in $(1 - \varepsilon_K, 1 + \varepsilon_K)$ and satisfying $\varepsilon_K \to_{K\to\infty} 0$, so that

$$\mathbb{P}(\ell(\eta^*_{N/K} - m_{N/K}) - g(K) \in [j-1, j)) = \mathbb{P}(\wp Y \in [j-1, j)) .$$

Because of (101), it suffices to prove (103) with $\ell(\eta^*_{N/K} - m_{N/K})$ in place of $\eta^*_{N/K} - m_{N/K}$.

We restrict attention to a fixed interval $[j-1, j)$. Introduce a linear scaling $\bar{\ell}$ of the variable v^* (again, with Lipshitz constant in $(1 - \varepsilon'_K, 1 + \varepsilon'_K)$, satisfying $\varepsilon'_K \to_{K\to\infty} 0$) so that

$$\mathbb{P}(\ell(\eta^*_{N/K} - m_{N/K}) - g(K) \in [j-1, j), K\bar{\ell}(v^*)/N \in \mathcal{W}^\delta) = \mathbb{P}(Y \in [j-1, j), Kz^K \in \mathcal{W}^\delta) .$$

($\bar{\ell}$ serves to shrink or expand $\mathcal{W}^\delta$ slightly.) Let μ^j_g and μ^j_c denote the probability measures on $[j-1, j) \times \mathcal{W}^\delta$ defined by

$$\begin{aligned}
\mu^j_g(I_1 \times I_2) &= \frac{\mathbb{P}(\ell(\eta^*_{N/K} - m_{N/K}) - g(K) \in I_1, Kv^*/N \in I_2)}{\mathbb{P}(\ell(\eta^*_{N/K} - m_{N/K}) - g(K) \in [j-1, j), K\ell'(v^*)/N \in \mathcal{W}^\delta)}, \\
\mu^j_c(I_1 \times I_2) &= \frac{\mathbb{P}(Y \in I_1, Kz^K \in I_2)}{\mathbb{P}(Y \in [j-1, j), Kz^K \in \mathcal{W}^\delta)},
\end{aligned}$$

for intervals I_1 and I_2. Note that μ^j_c has a positive density on $[j-1, j) \times \mathcal{W}^\delta$, which is uniformly bounded from below with a bound not depending on either j or K, and that the Lévy distance between μ^j_g and μ^j_c is bounded from above by $\varepsilon''_K \to_{K\to\infty} 0$, due to (11). Since $[j-1, j) \times \mathcal{W}^\delta$ is a bounded subset of $\mathbb{R}^3$, an elementary coupling (see, e.g., [4, Theorem 1.2]; the analog for one-dimensional couplings is easy to check) yields a coupling satisfying the analog of (103), but restricted to $[j-1, j)$. The claim (103) then follows by combining the couplings for different j. Further details are omitted. □

6.2 A continuity lemma

We will also need the following continuity result, which shows that the maximum value of the GFF is not affected by slightly changing the position at which the coarse field is sampled. In what follows, let z_i be as in Proposition 5.2, let $\bar{\varepsilon}_K \to_{K\to\infty} 0$ be as in Proposition 6.3, and let $\{z'_i\}_{i=1}^{K^2}$ denote a family of independent random variables chosen so that z'_i is measurable with respect to $\sigma(X^f_v, v \in V^{K,i}_N)$ and that satisfies $K|z_i - z'_i|/N \leqslant \bar{\varepsilon}_K$. (Recall that $\{X^f_v, v \in V^{K,i}_N\}$ are independent for distinct i.)

Lemma 6.4. *With notation as above,*

$$\limsup_{K\to\infty} \limsup_{N\to\infty} d(\max_{i=1}^{K^2}(X^f_{z_i} + X^c_{z_i}), \max_{i=1}^{K^2}(X^f_{z_i} + X^c_{z'_i})) = 0 . \tag{104}$$

Proof. The argument is similar to that for $\mathbb{P}(\mathcal{A}_2) \to_{N\to\infty} 0$, which was employed while proving Proposition 5.2. Denote by $V_{N,K}^{\times 2} = \{(u,v) : u,v \in V_N, |u-v| \leqslant \bar{\varepsilon}_K N/K\}$. For $(u,v) \in V_{N,K}^{\times 2}$, set $\zeta_{u,v,N,K} = \eta_{u,N} + X_u^c - X_v^c$ and $\zeta_{N,K}^* = \max_{(u,v)\in V_{N,K}^{\times 2}} \zeta_{u,v,N,K}$.

By Proposition 5.1 and (91) of Proposition 5.2, for given $\varepsilon > 0$,

$$|\max_{i=1}^{K^2}(X_{z_i}^f + X_{z_i}^c) - \eta_{N,K}^*| \leqslant \varepsilon$$

with probability $\to 1$, as first $N \to \infty$ and then $K \to \infty$. On the other hand, for z_i' as chosen above,

$$|\max_{i=1}^{K^2}(X_{z_i}^f + X_{z_i}^c) - \max_{i=1}^{K^2}(X_{z_i}^f + X_{z_i'}^c)| \leqslant \zeta_{N,K}^* - \eta_{N,K}^* .$$

It therefore follows from the definition of the Lévy distance that

$$\limsup_{K\to\infty} \limsup_{N\to\infty} d(\max_{i=1}^{K^2}(X_{z_i}^f + X_{z_i}^c), \max_{i=1}^{K^2}(X_{z_i}^f + X_{z_i'}^c)) \lesssim \mathbb{E}(\zeta_{N,K}^* - \eta_N^*) . \tag{105}$$

By arguments that are essentially identical to those in the proof of Lemma 5.4 (where we used Lemmas 3.5 and 3.10), $\mathbb{E}\zeta_{N,K}^* \leqslant \mathbb{E}(\max_{u\in V_N} \eta_{u,N} + \tilde{\varepsilon}_K \phi_{u,N})$, where $\tilde{\varepsilon}_K \to_{K\to\infty} 0$ and $\{\phi_{u,N}\}$ are independent variables satisfying

$$\mathbb{P}(\phi_{u,N} \geqslant 1 + \lambda) \leqslant \mathrm{e}^{-\lambda^2} \text{ for all } u \in V_N .$$

Application of Lemma 3.9 then implies $\mathbb{E}\zeta_{N,K}^* \leqslant \mathbb{E}\eta_N^* + C\sqrt{\tilde{\varepsilon}_K}$. Together with (105), this implies (104). □

6.3 Proof of Theorem 2.3

Fix $\varepsilon > 0$. Let $\alpha_{1,K,N}$ be as in Proposition 6.3 and recall that $|\alpha_{1,K,N} - g(K)| \leqslant \varepsilon_K$. Set

$$\bar{\eta}_N^* = \max_{\{i: X_{z_i}^f > m_{N/K} + g(K)\}} (X_{z_i}^f + X_{z_i}^c) .$$

By applying Proposition 5.1 together with (91) and (92) of Proposition 5.2, it follows, for fixed $\delta > 0$ and large enough K_0, that for each $K \geqslant K_0$ and large enough N,

$$\mathbb{P}(\eta_N^* > \bar{\eta}_N^* + \varepsilon) < \varepsilon .$$

Let $\nu_N^{K,\delta}$ denote the law of $\bar{\eta}_N^* - m_N$. Since $\eta_N^* \geqslant \bar{\eta}_N^*$, it follows from the definition of μ_N that $d(\mu_N, \nu_N^{K,\delta}) \leqslant \varepsilon$.

Set $X_i^{f,*} = \max_{v\in V_N^{K,\delta,i}} X_v^f$ and recall that $X_i^{f,*} = X_{z_i}^f$. Set $\bar{\wp}_i = 1_{\{X_i^{f,*} - m_{N/K} \geqslant \alpha_{1,K,N}\}}$, and let $\{\wp_i^K, Y_i^K, z_i^K\}_{i=1}^{K^2}$ be an i.i.d. sequence of random vectors, with $(\wp_i^K, Y_i^K, z_i^K)$ coupled to $(\bar{\wp}_i, X_i^{f,*} - m_{N/K}, z_i)$ as in Proposition 6.3. (Note that, for each N, the variables $(\wp_i^K, Y_i^K, z_i^K)$ require a different coupling; since their law does not depend N, we omit N from the notation.)

By Proposition 6.3, $K|z_i^K - z_i/N| \leqslant \bar{\varepsilon}_K$. Let $\bar{\nu}_N^{K,\delta}$ denote the law of $\max_{\{i:\wp_i^K = 1\}}(g(K) + Y_i^K + X_{z_i^K}^c - (m_N - m_{N/K}))$. It follows from Proposition 6.3 and Lemma 6.4 that, for some $K_1 \geqslant K_0$, $d(\bar{\nu}_N^{K,\delta}, \nu_N^{K,\delta}) < \varepsilon$ for each $K > K_1$ and large enough N.

Finally, by the convergence of X_N^c to $Z_{K,\delta}^c$, as $N \to \infty$, it follows that $d(\bar{\nu}_N^{K,\delta}, \mu_{K,\delta}) \to_{N\to\infty} 0$. We conclude that

$$\limsup_{K\to\infty} \limsup_{N\to\infty} d(\mu_N, \mu_{K,\delta}) < 2\varepsilon \, .$$

Letting $\varepsilon \to 0$ implies $\delta \to 0$, and hence demonstrates (13). Consequently, μ_N is a Cauchy sequence, which implies the existence of a limiting measure μ_∞ and completes the proof of the theorem.

□

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